

ON BASIC EMBEDDINGS INTO THE PLANE

DUŠAN REPOVŠ AND MATJAZŽ ŽELJKO

ABSTRACT. A subset $K \subset \mathbf{R}^2$ is said to be *basic* if for each function $f: K \rightarrow \mathbf{R}$ there exist functions $g, h: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. If all the three functions in this definition are assumed to be *continuous* (*differentiable*), then the embedding is C^0 -*basic* (C^1 -*basic*). This notion appeared in studies of Hilbert's 13th problem on superpositions. We prove that *if a finite graph is C^0 -basically embeddable in the plane, then it is C^1 -basically embeddable in the plane*. In our proof we construct an explicit C^1 -basic embedding and use the Skopenkov characterization of graphs C^0 -basically embeddable in the plane. Our result is nontrivial because the plane contains graphs which are C^0 -basic but not C^1 -basic and graphs which are C^1 -basic but not C^0 -basic (Baran-Skopenkov). We also prove that *given any integer $k \geq 0$, there is a subset of the plane which is C^r -basic for each $0 \leq r \leq k$ but not C^r -basic for each $k < r \leq \omega$* .

1. Introduction. The notion of a basic embedding appeared implicitly in the Kolmogorov-Arnold solution of Hilbert's 13th problem [1, 5, 6]. A compactum $K \subset \mathbf{R}^2$ is said to be *basic* if, for each continuous function $f: K \rightarrow \mathbf{R}$ there exist continuous functions $g, h: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$. One can replace in the definition of a basic embedding *continuous* functions by *smooth* functions (by Lipschitz, Hölder, analytic, etc., functions) and obtain a notion of basic embeddability in a smooth, Lipschitz, Hölder, analytic, etc. sense.

This note is motivated by the following problems.

Problem 1. *Find conditions on a compactum $K \subset \mathbf{R}^2$, under which K is basically embeddable into the plane in the smooth sense.*

2000 AMS *Mathematics subject classification*. Primary 54F50, 54C25, Secondary 46J10, 54C30.

Keywords and phrases. Basic embedding, linear relation, continuous function, array.

Received by the editors on November 27, 2003.

Problem 2. *Find conditions on a finite graph K , under which K is basically embeddable into the plane in the smooth sense.*

Problem 3. *Find conditions on an arbitrary compactum K , under which K is basically embeddable into the plane in the smooth sense.*

The answer to Problem 2 is given in the paper; the other two problems remain open.

For a subset K of the plane, not necessarily open, a function $f: K \rightarrow \mathbf{R}$ is said to be *r -analytic*, $0 \leq r < \infty$, if for each point $(x_0, y_0) \in K$ there exists

$$\{a_{ij}\}_{i,j=0}^r \subset \mathbf{R} \quad \text{such that} \quad a_{00} = f(x_0, y_0)$$

and

$$f(x_0 + x, y_0 + y) = \sum_{i,j=0}^r a_{ij} x^i y^j + o((|x| + |y|)^r),$$

where $(x_0 + x, y_0 + y) \in K$ and $(x, y) \rightarrow (0, 0)$. Since $\mathbf{R} \subset \mathbf{R}^2$, this definition applies to functions $\mathbf{R} \rightarrow \mathbf{R}$ as well. Note that 0-analytic is the same as continuous, 1-analytic for functions $\mathbf{R} \rightarrow \mathbf{R}$ is the same as differentiable and r -analytic for functions $\mathbf{R} \rightarrow \mathbf{R}$ is approximately (but not precisely) the same as C^r .

For a subset K of the plane (not necessarily open) a function $f: K \rightarrow \mathbf{R}$ is said to be *analytic* (or ω -analytic), if for each point $(x_0, y_0) \in K$ there exists

$$\{a_{ij}\}_{i,j=0}^{\infty} \subset \mathbf{R} \quad \text{such that} \quad f(x_0 + x, y_0 + y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$$

for $(x_0 + x, y_0 + y)$ belonging to some neighborhood of (x_0, y_0) in K .

A compactum $K \subset \mathbf{R}^2$ is said to be *C^r -basic*, $1 \leq r \leq \omega$, if for each r -analytic function $f: K \rightarrow \mathbf{R}$ there exist r -analytic functions $g, h: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

Theorem 1.1. *Given any integer $k \geq 0$, there is a subset of the plane which is C^r -basic for each $0 \leq r \leq k$ but not C^r -basic for each $k < r \leq \omega$.*

In Theorem 1.1 we can take *the graph* V_k of the function $y = |x|^k$, $x \in [-1, 1]$ for k odd, and $W_{k+1} = (V_{k+1} - (2, 0)) \sqcup (V_{k+1} + (2, 0))$ for k even.

The main result of this paper is the following.

Theorem 1.2. *If a finite graph K is C^0 -basically embeddable into the plane, then K is C^1 -basically embeddable into the plane.*

Theorem 1.2 is nontrivial because the plane contains graphs which are C^1 -basic but not C^0 -basic and graphs which are C^1 -basic but not C^0 -basic [3].

In the proof of Theorem 1.2 we use the following result, answering the Sternfeld problem [13].

Theorem 1.3 [11], cf. [7, 8], [10, Section 5]. *For any finite graph K the following conditions are equivalent:*

- (C) K is C^0 -basically embeddable in \mathbf{R}^2 ;
- (G) K does not contain any of the following three graphs: a circle S , a pentod P or a cross C with branched ends;
- (R) K can be embedded in R_n for some n .

Definition of the graphs R_n is given in Section 2. Our proof of Theorem 1.2 is based on a construction of a C^1 -basic embedding $R_n \subset \mathbf{R}^2$ (Section 2). We prove elementary that this embedding is also C^0 -basic, which yields an elementary proof of Theorem 1.3 as explained in Section 3.

2. Proofs.

Proof of Theorem 1.1 for k odd. First we prove that $V = V_1$ is C^1 -basic. Take a 1-analytic function $f: V \rightarrow \mathbf{R}$. Since f is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbf{R}$ such that

$$f(x, |x|) = f(0, 0) + ax + b|x| + o(|x| + |x|), \quad \text{where } x \rightarrow 0.$$

Take $h(y) = by$ and $g(x) = f(x, |x|) - h(|x|)$. Clearly, h is 1-analytic, i.e. differentiable, and g is 1-analytic outside 0. Since $g(x) = f(0, 0) + ax + o(x)$ when $x \rightarrow 0$, it follows that g is 1-analytic also at 0.

Now we prove that V_k is C^r -basic for each $0 \leq r \leq k$. Take an r -analytic function $f: V_k \rightarrow \mathbf{R}$. Since f is r -analytic at $(0, 0)$, it follows that there exists $\{a_{ij}\}_{i,j=0}^r \subset \mathbf{R}$ such that

$$a_{00} = f(0, 0) \quad \text{and} \quad f(x, |x|^k) = \sum_{i,j=0}^r a_{ij} x^i |x|^{kj} + o((|x| + |x|^r)^r),$$

where $x \rightarrow 0$. Since

$$o((|x| + |x|^r)^r) = o_1(x^r),$$

we have

$$f(x, |x|^k) = a_{00} + a_{01}|x|^k + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r).$$

Take $h(y) = a_{01}y$ and $g(x) = f(x, |x|^k) - h(|x|^k)$. Clearly, h is r -analytic and g is r -analytic outside 0. We also have $g(x) = a_{00} + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r)$ when $x \rightarrow 0$. So g is r -analytic also at 0.

Next we prove that $V = V_1$ is not C^r -basic for each $1 < r \leq \omega$. Define an analytic function $f: V \rightarrow \mathbf{R}$ by $f(x, y) = xy$, where $y = |x|$. If V is C^r -basic for some $r \geq 2$, then there are r -analytic functions

$$g, h: \mathbf{R} \rightarrow \mathbf{R} \quad \text{such that} \quad f(x, |x|) = x|x| = g(x) + h(|x|)$$

for each $x \in [0, 1]$. Hence $g(x) - g(-x) = 2x^2$. But this is impossible because g is 2-analytic, hence

$$g(x) = g(0) + ax + bx^2 + o(x^2) \quad \text{and so} \quad g(-x) = g(0) - ax + bx^2 + o(x^2)$$

for $x \rightarrow +0$.

At last we prove that V_k is not C^r -basic for k odd and each $k < r \leq \omega$. Define an analytic function $f: V_k \rightarrow \mathbf{R}$ by $f(x, y) = xy$, where $y = |x|^k$. If V is C^r -basic for some $r > k$, then there are r -analytic functions

$$g, h: \mathbf{R} \rightarrow \mathbf{R} \quad \text{such that} \quad f(x, |x|^k) = x|x|^k = g(x) + h(|x|^k)$$

for each $x \in [0, 1]$. Hence $g(x) - g(-x) = 2x|x|^k$. But this is impossible for k odd because g is $(k + 1)$ -analytic, hence

$$g(x) = g_0 + g_1x + \cdots + g_{k+1}x^{k+1} + o(x^{k+1})$$

and so

$$g(-x) = g_0 - g_1x + \cdots + g_{k+1}x^{k+1} + o(x^{k+1})$$

for $x \rightarrow +0$. \square

Note that a function $f(x, y)$ on the graph V is 1-analytic if and only if $p(t) = f(t, |t|)$ is differentiable on $[-1, 0]$ and on $[0, 1]$.

Proof of Theorem 1.1 for k even. Let us prove that W_{k+1} is C^r -basic for each $0 \leq r \leq k$. Given an r -analytic function $f: W_{k+1} \rightarrow \mathbf{R}$, take functions $h(y) = 0$ and $g(x) = f(x, |x - 2\text{sign } x|^{k+1})$. Clearly, h is r -analytic and $f(x, y) = g(x) + h(y)$ for each $(x, y) \in W_{k+1}$. Since the function $p(t) = |t|^{k+1}$ is k -analytic and $r \leq k$, it follows that g is r -analytic.

Let us prove that W_{k+1} is not C^r -basic for k even and each $k < r \leq \infty$. Define an analytic function $f: W_{k+1} \rightarrow \mathbf{R}$ by $f(x, y) = y\text{sign } x$. If W_{k+1} is C^r -basic, then there are r -analytic functions g and h such that $f(x, y) = g(x) + h(y)$.

For $x \in [-1, 1]$ we have

$$g(x - 2) + h(|x|^{k+1}) = f(x - 2, |x|^{k+1}) = -|x|^{k+1}$$

and

$$g(x + 2) + h(|x|^{k+1}) = f(x + 2, |x|^{k+1}) = |x|^{k+1}.$$

Hence $g(2 - x) = g(2 + x)$ and $g(-x - 2) = g(x - 2)$ for $x \in [-1, 1]$. Now $d^{k+1}g/dx^{k+1}|_{x=2} = d^{k+1}g/dx^{k+1}|_{x=-2} = 0$. This leads to a contradiction since g is $(k + 1)$ -analytic, $k + 1$ is odd, and $g(x + 2) - g(x - 2) = 2|x|^{k+1}$. \square

Let us define inductively the graphs R_n together with an embedding $R_n \rightarrow \mathbf{R}^2$. We embed R_1 into $[-10, 10] \times [-10, 10]$ as shown in Figure 1. Then we repeat the procedure by embedding copies of R_1 into squares A, B and C shown in Figure 1 to get R_2 . Note that the embedded R_1 into B was mirrored over ℓ to get a connected R_2 .

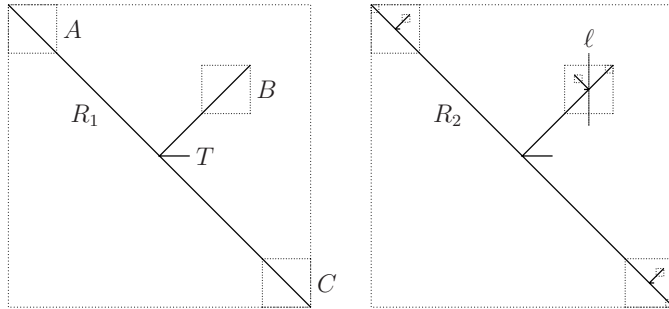


FIGURE 1.

In general, the graph R_n is constructed by embedding R_{n-1} into appropriate small squares A , B , C attached to R_1 . The squares A , B and C have to be chosen carefully. Let $p_1: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ and $p_2: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ denote projections onto x and y axes. We require that $p_1(A)$, $p_1(B)$, $p_1(C)$, $p_1(T)$ are disjoint and $p_2(A)$, $p_2(B)$, $p_2(C)$, $p_2(T)$ are disjoint.

Proof of Theorem 1.2. The boundary in R_n of any subgraph $K \subset R_n$ consists of a finite number of points. Hence any 1-analytic mapping $K \rightarrow \mathbf{R}$ can be extended to a 1-analytic mapping $R_n \rightarrow \mathbf{R}$. So it suffices to prove that R_n is C^1 -basic. We prove this by induction. Given a mapping $f: R_n \rightarrow \mathbf{R}$ we shall find functions $g, h: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$. Then we shall show that we can obtain g and h to be 1-analytic, i.e. differentiable, when f is 1-analytic.

Put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. Extend g to a function $g: [0, 10] \rightarrow \mathbf{R}$.

Note that for every $y \in [-10, 6]$ there exists a unique $x_y = |y| \in [0, 10]$ such that $(x_y, y) \in R_1$. (See Figure 2 for details.) Therefore, using g and f for $x \in [0, 10]$ we can define $h: [-10, 6] \rightarrow \mathbf{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Extend h to $h: [-10, 10] \rightarrow \mathbf{R}$.

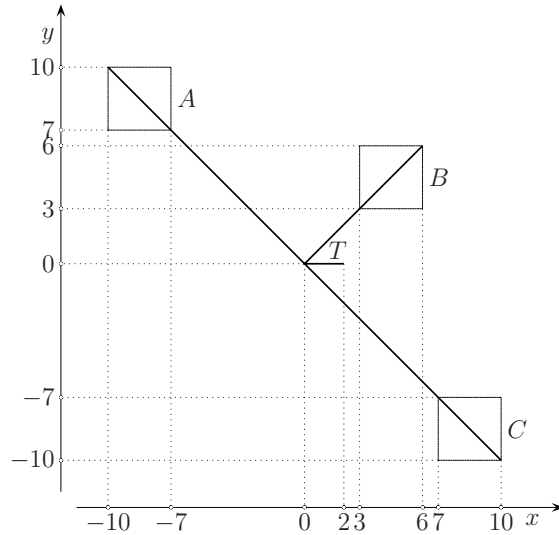


FIGURE 2.

Note that for every $x \in [-10, 0]$ there exists a unique $y_x = -x$ such that $(x, y_x) \in R_1$. Therefore using h we can define $g: [-10, 0] \rightarrow \mathbf{R}$ as $g(x) = f(x, -x) - h(-x)$. Finally, we extend g and h to $g, h: \mathbf{R} \rightarrow \mathbf{R}$.

Now let $f: R_n \rightarrow \mathbf{R}$, $n > 1$, be given. We put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. As R_n is constructed by embedding R_{n-1} into appropriate small squares A, B, C attached to R_1 , by inductive hypothesis there exist functions $g', h': \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g'(x) + h'(y)$ on $(x, y) \in (A \cup B \cup C) \cap R_n$. Hence we can extend g smoothly onto $[0, 10]$ so that $g = g'$ on $p_1(B \cup C)$. Using functions g and f for $x \in [0, 10]$ we can define $h: [-10, 6] \rightarrow \mathbf{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Then we extend h onto $[-10, 10]$ so that $h = h'$ on $[7, 10]$. Using h we finally define $g: [-10, 0] \rightarrow \mathbf{R}$ as $g(x) = f(x, -x) - h(-x)$.

For $n = 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable except maybe at 0. So all the extensions can be chosen to be differentiable. Since f is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbf{R}$ such that

$$f(x, y) = f(0, 0) + ax + by + o(|x| + |y|),$$

where $(x, y) \in R_1$ and $(x, y) \rightarrow (0, 0)$.

We may assume that $f(0, 0) = g(0) = h(0) = 0$. Then according to the structure of R_1 one can write

$$\begin{cases} f(x, x) = g(x) + h(x) \\ f(x, -x) = g(x) + h(-x) \\ f(x, 0) = g(x) \\ f(-x, x) = g(-x) + h(x), \end{cases}$$

so

$$\begin{cases} g(x) = f(x, 0) \\ h(x) = f(x, x) - f(x, 0) \\ h(-x) = f(x, -x) - f(x, 0) \\ g(-x) = f(-x, x) - f(x, x) + f(x, 0) \end{cases}$$

for small $x \geq 0$. Hence

$$g(x) = ax + o(x)$$

and

$$g(-x) = -ax + bx - ax - bx + ax + o(x) = -ax + o(x)$$

when $x \rightarrow +0$. So g is differentiable at 0. Also,

$$h(x) = ax + bx - ax + o(x) = bx + o(x)$$

and $h(-x) = ax - bx - ax + o(x) = -bx + o(x)$

when $x \rightarrow +0$. So h is differentiable at 0.

Hence, for $n > 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable everywhere. So all the extensions can be chosen to be differentiable and thus the resulting functions are differentiable. \square

An elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3. Analogously to the proof of Theorem 1.2 above. The reduction from K to R_n follows also by the Tietze-Uryhson extension theorem. We construct g and h from f as above. From the construction it is clear that at each step the constructed functions g and h are continuous. So all the extensions can be chosen to be continuous and thus the resulting functions are continuous. \square

Note that for each function $f: R_1 \rightarrow \mathbf{R}$ the functions $g, h: \mathbf{R} \rightarrow \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ are *uniquely* defined by f in a neighborhood of 0. Hence *any* such functions g and h are 0- or 1-analytic in a neighborhood of 0, if f is 0- or 1-analytic. Surprisingly, this is false for r -analytic functions with $1 < r \leq \omega$: the subset $R_1 \subset \mathbf{R}^2$ is C^1 -basic but not C^r -basic for each $1 < r \leq \omega$. This is proved analogously to Theorem 1.1 for k odd.

3. The Sternfeld criterion. The proof of Theorem 1.3 in [11] was based on the solution of the Arnold problem [2]: find conditions on a compactum $K \subset \mathbf{R}^2$, under which K is C-basic. This problem was solved by Sternfeld [12, 13] (who was apparently unaware of [2]). In order to formulate the Sternfeld criterion, let us introduce some definitions. Let p_1 and p_2 be projections onto the coordinate axes in \mathbf{R}^2 . For $Z \subset \mathbf{R}^2$, let

$$E(Z) = \{z \in Z : |Z \cap p_1^{-1}(p_1(z))| \geq 2 \quad \text{and} \quad |Z \cap p_2^{-1}(p_2(z))| \geq 2\}.$$

Set $E^2(Z) = E(E(Z))$, $E^3(Z) = E(E(E(Z)))$, etc. An ordered sequence $\{a_1, \dots, a_n\} \subset \mathbf{R}^2$ is called an *array* if, for each i , we have $p_1(a_i) = p_1(a_{i+1})$ for i even and $p_2(a_i) = p_2(a_{i+1})$ for i odd ($a_i \neq a_{i+1}$, but it is not required that all the points of an array should be distinct).

Theorem 3.1 [12, 13]. *For any compactum $K \subset \mathbf{R}^2$ the following conditions are equivalent:*

- (B) *the embedding $K \subset \mathbf{R}^2$ is basic;*
- (E) *$E^n(K) = \emptyset$ for some n ;*
- (A) *K does not contain any array of n points for some n .*

In this paper we prove Theorem 3.1 following [13] (we believe our exposition is clearer). One can see that the proof of Theorem 3.1 is non-elementary in a sense that it used the Banach inverse operator theorem.

The proof of $(R) \Leftrightarrow (G)$ in Theorem 1.3 is elementary, cf. [4]. The proof of $(C) \Rightarrow (G)$ in Theorem 1.3 is elementary modulo the implication $(B) \Rightarrow (A)$ of Theorem 3.1 [11]. The latter implication has an elementary proof by [9]. The proof of $(R) \Rightarrow (C)$ in Theorem 1.3 used the non-elementary implication $(E) \Rightarrow (B)$ of Theorem 3.1 [11]. In this paper we give an elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3, which yields an elementary proof of the whole Theorem 1.3.

The Sternfeld proof of Theorem 3.1. First we prove the easy assertion $(A) \Rightarrow (E)$. Suppose to the contrary that $E^n(K) \neq \emptyset$. Take a point $a_0 \in E^n(K)$. Then there exist points $a_{-1}, a_1 \in E^{n-1}(K)$ such that $p_1(a_{-1}) = p_1(a_0)$ and $p_2(a_1) = p_2(a_0)$. Analogously, there exist points $a_{-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n + 1$ points in K .

The proof of $(E) \Rightarrow (\Phi) \Rightarrow (A)$ is based on a reformulation of (B) terms of linear operators in functional spaces. Denote by $C(X)$ the space of continuous functions on X with the norm $|f| = \sup\{|f(x)| : x \in X\}$. For a subset $K \subset I^2$ define the *linear superposition operator*

$$\phi: C(I) \oplus C(I) \rightarrow C(K) \quad \text{by} \quad \phi(g, h)(x, y) = g(x) + h(y).$$

Clearly, the embedding $K \subset I^2$ is basic if and only if $\phi = \phi_K$ is epimorphic. Denote by $C^*(X)$ the space of bounded linear functionals on $C(X)$ with the norm $|\mu| = \sup\{|\mu(f)| : f \in C(X), |f| = 1\}$. For a subset $K \subset I^2$ define the *dual linear superposition operator*

$$\phi^*: C^*(K) \rightarrow C^*(I) \oplus C^*(I) \quad \text{by} \quad \phi^*\mu(g, h) = (\mu(g \circ p_1), \mu(h \circ p_2)).$$

Since $|\phi^*\mu| \leq 2|\mu|$, it follows that ϕ^* is bounded. By duality, ϕ_K is epimorphic if and only if $\phi^* = \phi_K^*$ is monomorphic. By the Banach inverse operator theorem, ϕ^* is monomorphic if and only if

$$(\Phi) \text{ there exists } \varepsilon > 0 \text{ such that } |\phi^*\mu| > \varepsilon|\mu| \text{ for each } \mu \in C^*(K)$$

(because this condition ensures that $\text{im } \phi^*$ is closed). Thus $(B) \Leftrightarrow (\Phi)$. So it remains to prove $(E) \Rightarrow (\Phi) \Rightarrow (A)$.

First we prove $(\Phi) \Rightarrow (A)$. If (A) is false, then for each n there exists an array $\{a_1, \dots, a_n\} \subset K$. Define a linear functional $\mu \in C^*(K)$ by $\mu(f) = \sum_{i=1}^n (-1)^i f(a_i)$. Then $|\mu| = n$ and $|\phi^* \mu| \leq 4$. Hence (Φ) is false.

Now we prove $(E) \Rightarrow (\Phi)$. We use the fact that $C^*(X)$ is the space of σ -additive regular real valued Borel measures (in the sequel - simply 'measures') on X . We have

$$\phi^* \mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(p_1^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(p_2^{-1}U).$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure μ to its positive and negative parts, then $|\mu| = \bar{\mu}(X)$, where $\bar{\mu} = \mu^+ + \mu^-$ is the absolute value of μ . Let D_x (D_y) be the set of points of K which are not shadowed by some other point of K in x - (y -) direction. Take any measure μ on K of the norm 1.

If

$$E(K) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K, \quad \text{so} \quad 1 = \bar{\mu}(K) \leq \bar{\mu}(D_x) + \bar{\mu}(D_y).$$

Therefore without loss of generality, $\bar{\mu}(D_x) \geq 1/2$. Since p_1 is injective over D_x , it follows that $|\mu_x| \geq 1/2$, thus (Φ) holds.

If

$$E(E(K)) = \emptyset, \quad \text{then} \quad D_x \cup D_y = K - E(K), \quad \text{so} \quad E(D_x \cup D_y) = \emptyset.$$

Therefore in the case when $\bar{\mu}(E(K)) < 3/4$ we have $\bar{\mu}(D_x \cup D_y) > 1/4$ and without loss of generality $\bar{\mu}(D_x) > 1/8$. Then as above $|\mu_x| > 1/8$, thus (Φ) holds. In the case when $\bar{\mu}(E(K)) \geq 3/4$ we have $\bar{\mu}(K - E(K)) \leq 1/4$. By the case $E(K) = \emptyset$ above without loss of generality $\bar{\mu}_x(p_1(E(K))) \geq \bar{\mu}(E(K))/2$. Hence $|\mu_x| \geq 1/2 \cdot 3/4 - 1/4 = 1/8$, thus (Φ) holds. The case of arbitrary n is proved analogously. \square

We remark that not only some linear relation on $\text{im } \phi_K$ can force it to be strictly less than $C(K)$. Or, in other words, φ_K^* can be injective but not monomorphic. If an embedding $K \subset \mathbf{R}^2$ is basic, then we can prove that ϕ^* is monomorphic without use of ϕ as follows. Define a linear operator

$$\Psi: C^*(I) \oplus C^*(I) \rightarrow C^*(K) \quad \text{by} \quad \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x, y) = g(x) + h(y)$ for $(x, y) \in K$. Clearly, $\Psi\Phi = \text{id}$ and Ψ is bounded, hence Φ is monomorphic.

Acknowledgments. Authors were supported in part by the Ministry for Education, Science and Sport of the Republic of Slovenia Research Program No. 101-509. We acknowledge Arkadiy Skopenkov for many useful discussions and suggestions and Jože Malešič, Neža Mramor-Kosta and Petar Pavešić for some discussions considering smooth basic embeddability. The authors also thank the referee for many useful comments that have helped to improve the content of this publication.

REFERENCES

1. V.I. Arnold, *On functions of three variables*, DAN SSSR **114** (1957), 679–681 (in Russian).
2. ———, *Problem 6 Math. Education* **3** (1958), 273 (in Russian).
3. M. Baran and A.B. Skopenkov, private communication.
4. A. Cavicchioli, D. Repovš and A.B. Skopenkov, *Open problems on graphs, arising from geometric topology*, Topol. Appl. **84** (1998), 207–226.
5. A.N. Kolmogorov, *On the representation of continuous functions of several variables by superposition of continuous functions of fewer variables*, DAN SSSR **108** (1956) (in Russian).
6. ———, *On the representation of continuous functions of several variables by superposition of functions of one variable and addition*, DAN SSSR **114** (1957), 953–956 (in Russian).
7. V.A. Kurlin, *Basic embeddings into products of graphs*, Topol. Appl. **102** (2000), 113–137.
8. ———, *Basic embeddings of graphs and the Dynnikov method of three-pages embeddings* Uspekhi Mat. Nauk **58** (2000), 163–164 (in Russian); Russian Math. Surveys **58** (2003), (in English).
9. N. Mramor-Kosta and E. Trenklerova, *On basic embeddings of compacta into the plane*, Bull. Austral. Math. Soc. **68** (2003), 471–480.
10. D. Repovš and A. Skopenkov, *New results on embeddings of polyhedra and manifolds into Euclidean spaces*, Uspekhi Mat. Nauk **54** (1999), 61–109 (in Russian); Russ. Math. Surv. 1149–1196 (in English).
11. A. Skopenkov, *A description of continua basically embeddable in \mathbf{R}^2* , Topol. Appl. **65** (1995), 29–48.
12. Y. Sternfeld, *Dimension, superposition of functions and separation of points in compact metric spaces*, Israel J. Math. **50** (1985), 13–53.
13. ———, *Hilbert's 13th problem and dimension*, Springer-Verlag, Berlin, 1989.

INSTITUTE FOR MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, P.O. Box 2964, 1001 LJUBLJANA, SLOVENIA
Email address: dusan.repovs@uni-lj.si

INSTITUTE FOR MATHEMATICS, PHYSICS AND MECHANICS, UNIVERSITY OF LJUBLJANA, P.O. Box 2964, 1001 LJUBLJANA, SLOVENIA
Email address: matjaz.zeljko@fmf.uni-lj.si