On defining sequences for Cantor sets

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Abstract

Each Cantor set can have many essentialy different defining sequences and there is no canonical way to choose one. The natural approach of reducing the number of handles does not work if the complement of a given Cantor set is too nice (for example: simply connected).

It is shown that there are no "unnecessary handles" in the defining sequence of a given Cantor if and only if the Cantor set satisfies some weakened 1-ULC property.

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1 Introduction

A defining sequence for a Cantor set $X \subset S^3$ (the 3-sphere) is a sequence (M_i) of compact 3-manifolds M_i with boundary such that (a) each M_i consists of disjoint cubes with handles (b) $M_{i+1} \subset \operatorname{Int} M_i$ for each i and (c) $X = \bigcap_i M_i$. We will always assume that M_0 is 3-ball which contains X in its interior.

It is known (see [1]) that every Cantor set has a defining sequence. In fact every Cantor set has many nonequivalent (see [6] for definition) defining sequences. One can expect if there are no cubes with "unnecessary handles" in the defining sequence then this sequence should carry much information about the embedding of this Cantor set in S^3.

2 Incompressible defining sequence

Let us formalize the term "unnecessary handles". A defining sequence (M_i) for a Cantor set $X \subset S^3$ is *incompressible* if M_0 is 3-ball and for every i each boundary component of $M_i \setminus \operatorname{Int} M_{i+1}$ is incompressible in $M_i \setminus \operatorname{Int} M_{i+1}$.

As R. Skora noticed in [7] the following lemma holds

Lemma 1 *If (M_i) is an incompressible defining sequence for a Cantor set X then every boundary component of every M_i is incompressible in $S^3 \setminus X$.*

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Bing proved in [3] that the Cantor set $X \subset S^3$ is tame if and only if there exists a defining sequence for X so the each M_i consists of a disjoint union of 3-balls only. Therefore the Cantor set X is tame if and only if each incompressible defining sequence for X consists of disjoint unions of 3-balls only.

We say that a Cantor set $X \subset S^3$ is *completly splittable* if any two distinct points of X can be separated by a 2-sphere in $S^3 \setminus X$. Every tame Cantor set and moreover every Cantor set with simply connected complement is completly splittable. A Bing Cantor set (see [2] for definition) is completly splittable too but its complement is not simply connected.

Theorem 2 *Let $X \subset S^3$ be a completly splittable Cantor set. Then X is tame if and only if there exists an incompressible defining sequence for it.*

PROOF. A tame Cantor set is obviously endowed with an incompressible defining sequence.

Let (M_i) be an incompressible defining sequence for X. By [3, Theorem 1.1] it sufficies to prove that X can be covered by a finite collection of arbitrarly small 3-balls.

Fix arbitrarly M_i, some component M of M_i and distinct points $x, y \in \operatorname{Int} M$. By assumption the points x and y can be separated by a 2-sphere S. We may assume that the sphere S lies entirely in M. As this can be done for every pair of distinct points in M we proceed as in proof of [3, Theorem 3.1]. The disjoint disks obtained in this way lie in M and hence they are arbitrarly small.

It remains only to explain why the sphere S may be assumed to lie in $\operatorname{Int} M$. Let the sphere S which separates x and y be a boundary of some 3-ball B. The sphere may be chosen to intersect the boundary of M (denoted by $\operatorname{Fr} M$) transversaly. If $S \cap \operatorname{Fr} M = \emptyset$ then S lies in the interior of M (denoted by $\operatorname{Int} M$). Otherwise we pick (not necessary unique) an innermost (with respect to S) component of $S \cap \operatorname{Fr} M$. This 1-sphere J bounds a 2-disk D on S. Note that $\operatorname{Int} D \cap M = \emptyset$ by the choice of J. By lemma 1 J bounds a 2-disk (say E) on $\operatorname{Fr} M$ because M is a component of an incompressible defining sequence. Furthermore $D \cup E$ is a 2-sphere because $D \cap E = \operatorname{Fr} D \cap \operatorname{Fr} E = J$ and hence it bounds two 3-balls in S^3. There are two posibilities:

- Let $\operatorname{Int} D \subset \operatorname{Int} M$. There exists one (say B') of the 3-balls bounded by $D \cup M$ such that $\operatorname{Int} B' \subset \operatorname{Int} M$. With respect to the position of x and y we have to consider three cases:
  - If only one of the points x or y lies in B' then $\operatorname{Fr} B'$ can be moved slightly to a sphere in $\operatorname{Int} M$ which separates these two points.
  - If $x, y \in B'$ then $\operatorname{Int} B' \cap S \neq \emptyset$. The (not necessarly unique) innermost (with respect to E) component of $S \cap \operatorname{Int} E$ bounds some 2-disk $E' \subset E$. As $\operatorname{Int} E' \cap S = \emptyset$, we consider two cases. If $\operatorname{Int} E' \subset \operatorname{Int} B$ we cut B along E' to obtain two 3-balls and exactly one of them (denoted by B'') contains one of the points x or y in its interior. If $\operatorname{Int} E' \subset S^3 \setminus B$ we attach 2-handle with core E' to B to obtain two nested 3-balls. As in previous case we denote by B'' the one of them which contains one of the points x or y in its interior.

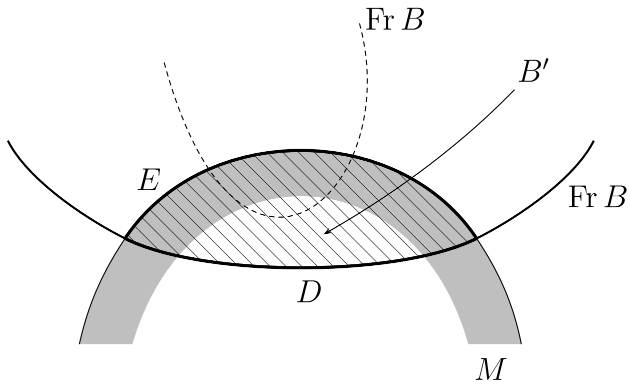

Figure 1: Case $\operatorname{Int} D \subset \operatorname{Int} M$

In both cases the number of components of $\operatorname{Fr} B'' \cap \operatorname{Fr} M$ is less than the number of components of $S \cap \operatorname{Fr} M$ so the procedure can be repeated with $\operatorname{Fr} B''$ and M.

– If neither x nor y lies in B' we cut B' from M and repeat the procedure with (new) M and the same sphere S.

- Let $\operatorname{Int} D \subset S^3 \setminus M$. Now we choose that one (say B') of the 3-balls bounded by $D \cup M$ such that $\operatorname{Int} B' \cap M = \emptyset$. Then points $x, y \in \operatorname{Int} M$ do not lie in B'. We may assume that $\operatorname{Int} E \cap \operatorname{Fr} B = \emptyset$. (If necessary we push $\operatorname{Fr} B$ out of B' into that part of slightly thickened disk B' which lies in M.) Now we consider two cases:

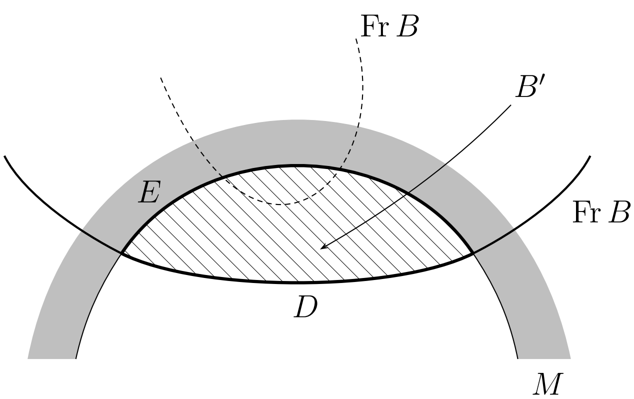

Figure 2: Case $\operatorname{Int} D \subset S^3 \setminus M$

– If $\operatorname{Int} E \subset \operatorname{Int} B$ then we cut B' out of B and repeat the procedure with M and diminuished disk B.

– If $\operatorname{Int} E \subset S^3 \setminus B$ then we attach B' to B and repeat the procedure with M and enlarged disk B.

3 Weakly 1-ULC property

We say that the complement of a Cantor set $X \subset S^3$ is 1-ULC if for each positive number ε there is a positive number δ such that every loop of diameter less than δ in $S^3 \setminus X$ is null-homotopic in $S^3 \setminus X$ in a set of diameter less than ε. By [3] or [4] we know that a Cantor set X is tame if and only if its complement has a property 1-ULC.

It is easy to see that the complement of a Cantor set $X \subset S^3$ is 1-ULC if and only if for any $x \in X$ and $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that for any map $f\colon S^1 \to \operatorname{Int} B(x, \delta) \setminus X$ there exists a map $F\colon B^2 \to \operatorname{Int} B(x, \varepsilon) \setminus X$ such that $F|_{\operatorname{Fr} B^2} = f$.

Let us modify the property 1-ULC slightly. We say the Cantor set $X \subset S^3$ is *weakly 1-ULC* if for any $x \in X$ and $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that for any map $f\colon S^1 \to \operatorname{Int} B(x, \delta) \setminus X$ the following holds: if $[f] = 0 \in \pi_1(S^3 \setminus X)$ then $[f] = 0 \in \pi_1(B(x, \varepsilon) \setminus X)$.

Every tame Cantor set is weakly 1-ULC and we leave the reader to prove that Antoine's necklace is weakly 1-ULC too.

Theorem 3 *Let $X \subset S^3$ be a Cantor set. There exists an incompressible defining sequence for X if and only if X is weakly 1-ULC.*

Proof. We assume that X is weakly 1-ULC. The incompressible defining sequence will be constructed inductively. Choose any 3-ball M_0 which contains X in its interior.

Assume now we have nested manifolds $M_0, \ldots, M_{i-1}$. Let $\varepsilon = \min\{\operatorname{dist}(X, \operatorname{Fr} M_{i-1}), 1/i\}$. By weakly 1-ULC property we pick for each $x \in X$ the corresponding $\delta_x < \varepsilon$ and finally we choose the Lebesgue δ for the covering $\{\operatorname{Int} B(x, \delta_x);\ x \in X\}$ of X. There exists a manifold M_i whose components are cubes with handles of diameter less than δ. We choose such M_i that the Euler characteristic $\chi(\operatorname{Fr}(M_{i-1} \setminus \operatorname{Int} M_i))$ is minimal. Hence M_i has incompressible boundary components in $M_i \setminus X$.

If some boundary component of $M_{i-1} \setminus \operatorname{Int} M_i$ is compressible, then there exist a nontrivial loop J which bounds a 2-disk in $M_{i-1} \setminus \operatorname{Int} M_i$. By weakly 1-ULC property there exist a singular disk $f\colon B^2 \to N(J, \varepsilon) \setminus X$. Using the regular neighbourhood for $\operatorname{Fr} M_i$ in $M_{i-1} \setminus \operatorname{Int} M_i$ we modify f to become embedding near $\operatorname{Fr} B^2$. We may also assume that f is transversal to $\operatorname{Fr} M_i$ and that the number of components of $f^{-1}(\operatorname{Fr} M_i)$ is minimal.

The innermost component of $f^{-1}(\operatorname{Fr} M_i)$ bounds some 2-disk $D \subset B^2$ such that $f^{-1}(\operatorname{Fr} M_i) \cap D = \operatorname{Fr} D$. If $f(\operatorname{Fr} D) \simeq 0$ in $\operatorname{Fr} M_i$ we could modify f near D to reduce the number of components of $f^{-1}(\operatorname{Fr} M_i)$. Hence $f(\operatorname{Fr} D) \not\simeq 0$ and we also notice that $f(D) \subset M_{i-1} \setminus \operatorname{Int} M_i$ as $\operatorname{Fr} M_i$ being incompressible in $M_i \setminus X$. Using the regular neighbourhood for $\operatorname{Fr} M_i$ in $M_{i-1} \setminus \operatorname{Int} M_i$ we modify f appropriately and then use Dehn lemma to obtain a 2-disk $E \subset M_{i-1} \setminus \operatorname{Int} M_i$, $\operatorname{Fr} E = f(\operatorname{Fr} D)$, near $f(D)$.

Now we attach a 2-handle H with core E to M_i to obtain a manifold M_i'. It follows $\chi(\operatorname{Fr}(M_i')) < \chi(\operatorname{Fr}(M_i))$ and $\operatorname{diam}(M_i') < \varepsilon$. If the boundary of $M_{i-1} \setminus \operatorname{Int} M_i'$ is still compresible we find a loop $J' \subset \operatorname{Fr}(M_{i-1} \setminus \operatorname{Int} M_i')$ which bounds some singular disk in $M_{i-1} \setminus \operatorname{Int} M_i'$. As M_i' is obtained from M_i by attaching 2-handle we may assume that $J' \subset \operatorname{Fr} M_i \setminus H$. Therefore $\operatorname{diam}(J') < \delta$ and we can repeat the procedure. It terminates after finitely many steps because $\chi(\operatorname{Fr} M_i)$ decreases. At the end we obtain the manifold

M_i, $\mathrm{diam}(M_i) < \varepsilon$, such that boundary components of $M_{i-1} \setminus \operatorname{Int} M_i$ are incompressible. The inductive step is now proven.

The 'only' part is easier to prove. Let (M_i) be an incompressible defining sequence for X. Pick arbitrarly $x \in X$ and let $\varepsilon > 0$. Then there exists M_i that the component M of M_i which contains x has diameter less than ε. Define $\delta = \mathrm{dist}(x, \operatorname{Fr} M)$.

Let $f\colon S^1 \to B(x,\delta)\setminus X$ be such map that $[f] = 0 \in \pi_1(S^3 \setminus X)$. Then the map f can be extended to $F\colon B^2 \to S^3 \setminus X$. We may assume that $f(B^2) \cap \operatorname{Fr} M = \emptyset$ by incompressibility of boundary components of defining sequence and standard cut and paste argument. (One has to put f transversal to $\bigcup_i \operatorname{Fr} M_i$ and consider the innermost component of $f^{-1}(\bigcup_i \operatorname{Fr} M_i)$.) Hence the extension F lies entirely in M and $[f] = 0 \in \pi_1(B(x,\varepsilon) \setminus X)$ as required.

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