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GENUS OF A CANTOR SET

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ABSTRACT. We define a genus of a Cantor set as the minimal number of the maximal number of handles over all possible defining sequences for it. The relationship between the local and the global genus is studied for genus 0 and 1. The criterion for estimating local genus is proved along with the example of a Cantor set having prescribed genus. It is shown that some condition similar to 1-ULC implies local genus equal to 0.

1. Introduction. We will consider Cantor sets embedded in three-dimensional Euclidean space $\mathbb{E}^3$. A defining sequence for a Cantor set $X \subset \mathbb{E}^3$ is a sequence (M_i) of compact 3-manifolds M_i with boundary such that each M_i consists of disjoint cubes with handles, $M_{i+1} \subset \operatorname{Int} M_i$ for each i and $X = \cap_i M_i$. We denote the set of all defining sequences for X by $\mathcal{D}(X)$.

Armentrout [**1**] proved that every Cantor set has a defining sequence. In fact every Cantor set has many nonequivalent, see [**7**] for definition, defining sequences and in general there is no canonical way to choose one. One approach is to compress unnecessary handles in the given defining sequence for a Cantor set. A class for which this process terminates is characterized by some property similar to 1-ULC, see [**10**] for details. But in general this process is infinite so the "incompressible" defining sequence may not exist. Hence we look at the minimal number of the maximal number of handles over all possible defining sequences for it and take the defining sequence for which this number is minimal. Unfortunately this sequence need not to be canonical, but the minimal number, i.e. *the genus*, itself has some interesting properties.

Using different terminology Babich [**2**] actually proved that the genus of a wild scrawny, see [**2**] for definition, Cantor set is at least 2.

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2. The genus. Let M be a cube with handles. We denote the number of handles of M by $g(M)$. For a disjoint union of cubes with handles $M = \sqcup_{\lambda\in\Lambda} M_\lambda$, we define $g(M) = \sup\{g(M_\lambda);\ \lambda \in \Lambda\}$.

Let (M_i) be a defining sequence for a Cantor set $X \subset \mathbb{E}^3$. For any subset $A \subset X$ we denote by M_i^A the union of those components of M_i which intersect A. Define

$$g_A(X; (M_i)) = \sup\{g(M_i^A);\ i \geq 0\}$$

and

$$g_A(X) = \inf\{g_A(X; (M_i));\ (M_i) \in \mathcal{D}(X)\}.$$

The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A*. For $A = X$ we call the number $g_X(X)$ *the genus of the Cantor set X* and denote it simply by $g(X)$. For any point $x \in X$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

As a trivial consequence of the definition one can prove

Lemma 1. *Genus of a Cantor set is a monotone function. Precisely*:

1. *For $A \subset B \subset X$ where X is a Cantor set we have $g_A(X) \leq g_B(X)$.*

2. *For $A \subset X \subset Y$ where X is a closed subset of a Cantor set Y we have $g_A(X) \leq g_A(Y)$.*

By the standard construction of Antoine's necklace $\mathcal{A}$ we know $g(\mathcal{A}) \leq 1$. As the Cantor set $\mathcal{A}$ is wild we conclude $g(\mathcal{A}) = 1$. So there exists a Cantor set of genus 1. We call such Cantor sets *toroidal*.

Using the result of Babich [**2**] one can prove that there exists a Cantor set of genus 2. We will extend the theorem [**2**, Theorem 2] to obtain a criterion for estimating the local genus and thus constructing a Cantor set of arbitrary genus.

3. Genus 0. By a theorem of Bing [**4**] we know that the Cantor set $X \subset \mathbb{E}^3$ is tame if and only if $g(X) = 0$. By a theorem of Osborne [**5**, Theorem 4] we know that the Cantor set $X \subset \mathbb{E}^3$ is tame if and only if $g_x(X) = 0$ for every point $x \in X$.

Theorem 2. *Let x be an arbitrary point of a Cantor set $X \subset \mathbb{E}^3$. If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every mapping $f: S^1 \to \operatorname{Int} B(x, \delta) \setminus X$ there exists a map $F: B^2 \to \operatorname{Int} B(x, \varepsilon) \setminus X$ that $F|_{S^1} = f$ then $g_x(X) = 0$.*

Proof. It suffices to find a sequence of nested 3-balls M_i whose boundaries do not intersect X such that $\{x\} = \cap_i M_i$.

The sequence (M_i) will be constructed inductively. Let M_1 be some large 3-ball. Assume now that the 3-balls $M_1, M_2, \dots, M_k$ are constructed. Let $\varepsilon = \operatorname{dist}(x, \operatorname{Fr} M_k)/2$ and pick δ according to the hypothesis of the theorem. We may assume that $\delta < \varepsilon$.

There exists a cube with handles (denote this cube by M) of diameter at most $\delta/2$ which contains x in its interior and its boundary does not intersect X, see [**1**, Paragraph 7] for details. Let s be the number of handles of M. If $s = 0$ put $M_{k+1} := M$, and the inductive step is proven. If $s > 0$, let J be one of the meridional curves on $\operatorname{Fr} M$. By hypothesis of the theorem there exist a singular disk $f: B^2 \to \operatorname{Int} B(x, \varepsilon) \setminus X$ with boundary J. We can modify f near S^1 such that it embeds some small collar of S^1 in B^2 into some small collar of $\operatorname{Fr} M$ in $M \setminus X$. We may also assume that f is PL and transversal to $\operatorname{Fr} M$.

If $f^{-1}(\operatorname{Fr} M) \subset \operatorname{Int} B^2$ has at least one component we pick the innermost one and compress $\operatorname{Fr} M$ along 2-disk bounded by this component. (We either cut M along this disk or attach 2-handle onto M having this disk as a core.) If $f^{-1}(\operatorname{Fr} M) = \varnothing$ then $f(\operatorname{Int} B^2) \subset \operatorname{Int} M$. Hence $\operatorname{Fr} M$ is compressible in $M \setminus X$. Using the Loop theorem we find an appropriate compressing disk and reduce the number of handles in M.

If the cube with handles obtained in the previous step has some more handles we repeat the procedure. As it is possible that the new meridional curve J intersects some attached 2-handle we must push it off this handle to have the diameter of J small enough. This procedure stops after at most s steps. □

Remark. The reader may note that the hypothesis of this theorem is not enough for the Cantor set X to be locally tame at x. However if the hypothesis of the theorem is satisfied for every $x \in X$ we obtain the well known 1-ULC taming theorem due to Bing [**4**].

4. The existence of a Cantor set of arbitrary genus. Let Γ be a tree having $r+1$ nodes. For $k \in \{2,3,\dots,r\}$ we denote by $G(\Gamma,r,k)$ the number of nodes of Γ whose degree is at most k. We define

$$G(r,k) = \inf\{G(\Gamma,r,k);\ \Gamma \text{ is a tree with } r+1 \text{ nodes}\}.$$

Lemma 3. *Using the above notation we estimate*

$$\lceil r+1-(r-1)/k\rceil \le G(r,k) \le r+1,$$

where $\lceil x\rceil$ *denotes the least integer not less than given* $x \in \mathbb{R}$ *(for example* $\lceil \pi\rceil = 4$*).*

Proof. Let Γ be a an arbitrary tree having $r+1$ nodes. We denote by v_i the number of nodes of Γ whose degree is equal to i. Hence

$$v_1 + 2\,v_2 + \cdots + r\,v_r = 2r, \tag{1}$$

as every edge is counted twice. The tree Γ has $r+1$ nodes so

$$v_1 + v_2 + \cdots + v_r = r+1. \tag{2}$$

The number of nodes of Γ having degree at most k equals to

$$G(\Gamma,r,k) = v_1 + v_2 + \cdots + v_k.$$

We estimate

$$\begin{aligned}
2r &\overset{(1)}{=} v_1 + 2\,v_2 + \cdots + k\,v_k + (k+1)\,v_{k+1} + \cdots + r\,v_r \\
&\ge v_1 + 2\,v_2 + \cdots + k\,v_k + (k+1)(v_{k+1} + \cdots + v_r) \\
&\overset{(2)}{=} v_1 + 2\,v_2 + \cdots + k\,v_k + (k+1)((r+1) - (v_1 + \cdots + v_k)) \\
&= (k+1)(r+1) - (kv_1 + (k-1)v_2 + \cdots + v_k) \\
&\ge (k+1)(r+1) - k(v_1 + v_2 + \cdots + v_k),
\end{aligned}$$

and hence

$$G(\Gamma,k,r) = v_1 + v_2 + \cdots + v_k \ge r + 1 - \frac{1}{k}(r-1).$$

As $G(\Gamma, k, r)$ is integer we can sharpen the estimate $G(\Gamma, k, r) \geq \lceil r + 1 - (r-1)/k \rceil$ to get the required inequality. $\square$

Remark. For $k = 2$ we have $G(r, 2) \geq \lceil r + 3/2 \rceil$ and for $k = r$ we have $G(r, r) \geq \lceil r + 1/r \rceil = r + 1$.

Using the following criterion we can estimate the lower bound for local genus of a Cantor set.

Theorem 4. *Let $X \subset \mathbb{E}^3$ be a Cantor set and $x_0 \in X$ be an arbitrary point. Let there exist a 3-ball B and 2-disks $D_1, \ldots, D_r$ such that*

1. *For every disk D_i we have $D_i \cap X = \operatorname{Int} D_i \cap X = \{x_0\}$.*
2. *For distinct pair of disks D_i in D_j we have $D_i \cap D_j = \{x_0\}$.*
3. *The point x_0 lies in the interior of B and $\operatorname{Fr} D_i \cap B = \varnothing$ for every disk D_i.*
4. *If there exists a planar compact surface in $B \setminus X$ whose boundary components lie in $(D_1 \cup \cdots \cup D_r) \cap \operatorname{Fr} B$ then this surface has at least $k+1$ boundary components.*

Then $g_{x_0}(X) \geq G(r, k)$.

Proof. We will prove that every cube with handles $N \subset \operatorname{Int} B$ such that $x_0 \in N$ and $\operatorname{Fr} N \cap X = \varnothing$, has at least $G(r, k)$ handles. We may assume that D_i intersects $\operatorname{Fr} N$ transversally (shortly $D_i \pitchfork \operatorname{Fr} N$) and that $\operatorname{Fr} N$ has minimal genus. We may also assume that among all cubes with $g(\operatorname{Fr} N)$ handles N minimizes the number of components of $\operatorname{Fr} N \cap (D_1 \cup \cdots \cup D_r)$.

Fix disk D_i. The intersection $D_i \cap \operatorname{Fr} N$ has at least one component and each of them bounds a disk in $\operatorname{Int} D_i$. If some of such disks in $\operatorname{Int} D_i$ does not contain x_0 we pick the innermost one and denote it by E. (Disk E need not be unique.) The loop $\operatorname{Fr} E$ bounds a disk $E^* \subset \operatorname{Fr} N$ as otherwise N could be compressed along E and hence $g(\operatorname{Fr} N)$ would decrease. So we can replace E by E^* in order to decrease the number of components in $\operatorname{Fr} N \cap D_i$.

Therefore the components of $D_i \cap \operatorname{Fr} N$ are nested and each of them bounds a disk containing x_0. The number of components is odd as $x_0 \in D_i \cap N$ and $\operatorname{Fr} D_i \cap N = \varnothing$. If $D_i \cap \operatorname{Fr} N$ has at least three components there exist consecutive two of them which bound an annulus $A \subset D_i$ such that $A \cap \operatorname{Fr} N = \operatorname{Fr} A$ and $A \subset N$. Now we cut N along A to obtain the manifold N^* which has at most two components. As $\chi(A) = 0$ we have $\chi(\operatorname{Fr} N) = \chi(\operatorname{Fr} N^*)$. If N^* has two components we dispose of that one which does not contain x_0. Therefore $g(\operatorname{Fr} N^*) \leq g(\operatorname{Fr} N)$ and the number of components of $\operatorname{Fr} N^* \cap D_i$ is less than the number of components of $\operatorname{Fr} N \cap D_i$. We repeat the procedure until there is only one component of $\operatorname{Fr} N \cap D_i$ left. The remaining component, say η_i, separates $\operatorname{Fr} N$ as D_i separates N.

So there are exactly $r + 1$ components of $\operatorname{Fr} N \setminus (\eta_1 \cup \cdots \cup \eta_r)$. Let us denote their closures by $K_1, \dots, K_{r+1}$. For every i the compact surface K_i is either nonplanar having at least one boundary component or planar having at least $k + 1$ boundary components. The surface K_i cannot be a disk with less than k holes as otherwise one can attach onto it appropriate annuli in D_i bound by η_i and $\operatorname{Fr} B \cap D_i$ to obtain a planar surface in $B \setminus X$ having at most k boundary components (and all of them are contained in $(D_1 \cup \cdots \cup D_r) \cap \operatorname{Fr} B$).

Finally we construct a graph Γ related to the components of $\operatorname{Fr} N \setminus (\eta_1 \cup \cdots \cup \eta_r)$. The nodes of Γ shall be $\{K_1, \dots, K_{r+1}\}$. The nodes K_i and K_j are connected in Γ if and only if $K_i \cap K_j \neq \varnothing$. The graph Γ is a tree as each of $\eta_1, \dots, \eta_r$ separates $\operatorname{Fr} N$. The tree Γ has at least $G(r, k)$ nodes of degree at most k so there are at least $G(r, k)$ nonplanar components in $\{K_1, \dots, K_{r+1}\}$. Hence $g(\operatorname{Fr} N) \geq G(r, k)$. □

Remark. It is easier to check the last condition in the statement of the theorem when k is small but we get the most out of this criterion for $k = r$ as we have $G(r, r) = r + 1$.

Theorem 5. *For every number $r \in \mathbb{N} \cup \{0, \infty\}$ there exists a Cantor set $X \subset \mathbb{E}^3$ such that $g(X) = r$.*

Proof. For the sake of simplicity we replace $\mathbb{E}^3$ by S^3. We know that every tame Cantor set has genus 0 and for example the Antoine's necklace has genus 1. Therefore we may assume $2 \leq r < \infty$.

Fix arbitrary point $x_0 \in S^3$. We will construct a defining sequence (M_i) for the Cantor set X. Let M_1 be a cube with r handles containing

x_0 in its interior. The manifold M_2 shall have $5r+1$ components. One of them, denoted by M_2^0, is a cube with r handles containing x_0 in its interior. We link each handle of M_2^0 by a chain of five tori and this chain is spread along the core of some of the handles in M_1. Now we construct the manifold M_3. The components of M_3 which lie in toroidal components of M_2 for a chain of linked tori (use the Antoine construction) and there are $5r+1$ components of M_3 in M_2^0 embedded in the same way as M_2 is embedded in M_1. Repeat the procedure inductively. (See Figure 1 for details. There are only two "legs" of X drawn in the figure, the remaining $r-2$ ones are supposed to be in the dotted part in the middle.)

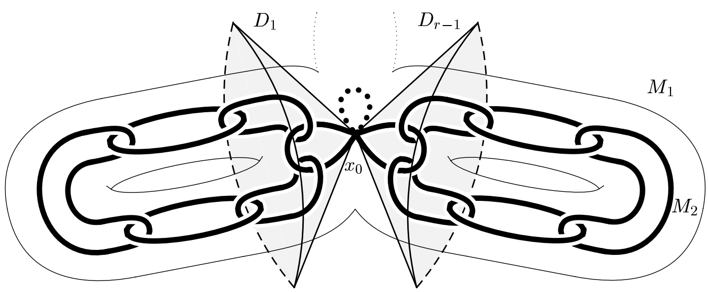

FIGURE 1. Defining sequence for a Cantor set of genus r, $r \geq 2$.

By construction it is clear that $g(X) \leq r$. Using the $r-1$ disks $D_1, \dots, D_{r-1}$ and the criterion 4 we will prove that $g_{x_0}(X) \geq r$.

We have to prove that there does not exist a planar surface $F \subset \operatorname{Int} B \setminus X$ which has r boundary components $\gamma_1, \dots, \gamma_r$ such that $\gamma_i \subset D_i$ and γ_i is parallel to $\operatorname{Fr} D_i$ in D_i. Assume to the contrary: let such F exist.

Simple connected curves γ_i bounds disks $E_i \subset \operatorname{Int} D_i$ and $x_0 \in \operatorname{Int} E_i$ for every i. By attaching disks E_i to the surface F we obtain a singular sphere Σ. As there are $r+1$ "legs" of Cantor set joining in x_0 but only r "peaks" in Σ there exists a point $a \in X$ close to x_0 such that $\operatorname{lk}_{\mathbb{Z}_2}(\Sigma, a) = 1$ (i.e. singular sphere Σ winds around a). Let A be the "leg" of X which contains a. Therefore A is a Cantor set obviously homeomorphic to the Antoine's necklace. The singular sphere Σ can be modified near x_0 so that it lies in $S^3 \setminus A$. (One has just to space out the peaks of Σ near x_0.) Let $f\colon S^2 \to \Sigma$ be a continuous map representing Σ. Let

$$h\colon \pi_2(S^3 \setminus A) \to H_2(S^3 \setminus A; \mathbb{Z})$$

be a Hurewicz homomorphism and

$$m\colon H_2(S^3 \setminus A; \mathbb{Z}) \to H_2(S^3 \setminus A; \mathbb{Z}_2)$$

be a map induced by homomorphism $\bmod 2\colon \mathbb{Z} \to \mathbb{Z}_2$. Kernel of a map h is a subgroup of $\pi_2(S^3 \setminus A)$ which we denote by N. If $[f] \in N$ then also $mh([f]) = 0 \in H_2(S^3 \setminus A; \mathbb{Z}_2)$ but this contradicts $\operatorname{lk}_{\mathbb{Z}_2}(\Sigma, a) = 1$. Hence $[f] \notin N$. Using the sphere theorem we replace f by a nonsingular sphere $g\colon S^2 \to S^3 \setminus X$. As $[g] \neq 0 \in \pi_2(S^3 \setminus X)$ the sphere $g(S^2)$ winds around at least one point of A, but not around all of them. Therefore some two points of A can be separated by sphere in $S^3 \setminus A$. But it is well known that this is impossible. Hence by Theorem 4 we have $g_{x_0}(X) \geq r$ and therefore $g(X) = r$.

Finally we prove the case $r = \infty$. Let X_r be a Cantor set of genus $r \in \mathbb{N}$. One can take a disjoint union of X_rs converging to the point, say x_∞. Therefore $X = \sqcup_r X_r$ is a Cantor set and $g_{x_\infty}(X) = \infty = g(X)$. □

Remark. The Cantor set in the previous theorem does not have simply connected complement (except for $r = 0$). It is interesting to note that, using the same construction, one can exhibit a Cantor set of arbitrary genus with simply connected complement. We just have to replace the building block: instead of Antoine's necklace we use Bing-Whitehead Cantor set as its complement is simply connected, see [**9**] for details. The proof itself is almost the same: for the final contradiction we refer to [**3**, Paragraph 5] as Bing-Whitehead Cantor set can be separated by spheres but not with arbitrarily small ones.

Let $X \subset \mathbb{E}^3$ be a Cantor set. From 1 we see that $g_x(X) \leq g(X)$ for every point $x \in X$. The author believes that the following conjecture may not be true in general:

Conjecture 1. *For every Cantor set X there exists a point $x \in X$ such that $g_x(X) = g(X)$.*

The conjecture may be restated as

Conjecture 2. *Let $g_x(X) \leq r$ for every point x of a Cantor set X. Then $g(X) \leq r$.*

For $r = 0$, however, this is true [**5**]. We will prove this conjecture for $r = 1$ under some additional technical hypothesis.

5. Local genus versus global genus. Let $X \subset \mathbb{E}^3$ be a Cantor set. We say that the Cantor set X is *splittable* if there exists a 2-sphere S in the complement of X which separates some two points of X. For a splittable Cantor set we may define $\mu(X) = \inf\{\operatorname{diam}(S);\ S \in \mathcal{S}\}$ where $\mathcal{S}$ is a set of separating 2-spheres for X. If a Cantor set X is not splittable we set $\mu(X) = \infty$. The number $\mu(X)$ is called *the lower bound of splittability.*

The number $\mu(X)$ certainly depends on embedding $X \hookrightarrow \mathbb{E}^3$. One can prove that for equivalently embedded, see [**7**] for definition, Cantor sets X and X' we have

$$\mu(X) = 0 \text{ if and only if } \mu(X') = 0,$$
$$\mu(X) > 0 \text{ if and only if } \mu(X') > 0,$$
$$\mu(X) = \infty \text{ if and only if } \mu(X') = \infty.$$

Obviously $\mu(X) = 0$ for a tame Cantor set X. One can easily construct a wild Cantor set X such that $\mu(X) = 0$. As the Antoine's necklace $\mathcal{A}$ is not splittable we have $\mu(\mathcal{A}) = \infty$. Finally there exists a wild cantor set with positive lower bound of splittability, see [**3**, p. 361] for more details.

Lemma 6. *Let $\mu(X) > 0$ for a given Cantor set $X \subset \mathbb{E}^3$. Let M and N be two solid tori in $\mathbb{E}^3$ such that $\operatorname{Fr} M \pitchfork \operatorname{Fr} N$, $X \subset M \cup N \setminus (\operatorname{Fr} M \cup \operatorname{Fr} N)$ and $\operatorname{diam}(M \cup N) < \mu(X)$. Then for every $\eta > 0$ there exist (at most) two disjoint solid tori whose interiors cover X and each of them lies entirely in $\{x \in \mathbb{E}^3;\ \operatorname{dist}(x, M) < \eta\}$ or $\{x \in \mathbb{E}^3;\ \operatorname{dist}(x, N) < \eta\}$.*

Proof. Denote $\mu(X)$ simply by μ. We may assume that $\operatorname{diam}(M \cup N) + \eta < \mu$. As $\operatorname{Fr} M \pitchfork \operatorname{Fr} N$ the components of $\operatorname{Fr} M \cap \operatorname{Fr} N$ are 1-spheres and the proof will be done by induction on the number of components in $\operatorname{Fr} M \cap \operatorname{Fr} N$. Case $\operatorname{Fr} M \cap \operatorname{Fr} N = \varnothing$ is obvious.

If $\operatorname{Fr} M \cap \operatorname{Fr} N \neq \varnothing$ we distinguish three cases. If some component of $\operatorname{Fr} M \cap \operatorname{Fr} N$ bounds a 2-disk, say on $\operatorname{Fr} M$ by symmetry, we pick an innermost of such components, with respect to $\operatorname{Fr} M$, and denote it by J. Then $J = \operatorname{Fr} D$ for some 2-disk D.

Trivial case. The loop J is not contractible on $\operatorname{Fr} N$ so D is a compressing disk for N. Then we can cut N along D or attach 2-handle with core D onto N and obtain a 3-ball. As this disk is small enough it contains either whole X or it is disjoint to X. Then either M or N is unnecessary.

The 3*-ball case.* The loop bounds some 2-disk E on $\operatorname{Fr} N$ and therefore $D \cup E = \operatorname{Fr} B$ for some 3-ball B. Now we analyze two subcases:

• *Inner disk D*, see Figure 2: The 3-ball B lies in N and is disjoint to X, or contains X which is trivial–the torus M can be disposed.

As we cut out B from N along disk D we obtain a torus $N^* \subset N$. The number of components of $\operatorname{Fr} \cap \operatorname{Fr} M$ is less than the number of components of $\operatorname{Fr} N \cup \operatorname{Fr} M$. We conclude the proof using induction hypothesis on solid tori M and N^*.

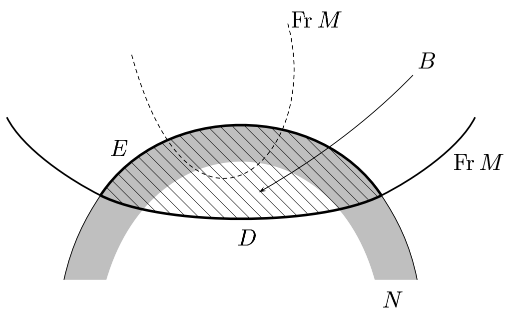

FIGURE 2. Inner disk D with respect to N.

• *Outer disk* D, see Figure 3: The 3-ball B does not lie in N. If B lies in M, then either $X \subset B$ (hence N can be disposed) or $X \cap B = \varnothing$. If $B \cap M = \varnothing$ then certainly $X \cap B = \varnothing$. Therefore we may assume $B \cap X = \varnothing$. There exists some η', $0 < \eta' < \eta$, such that the η'-neighborhood of B does not intersect X.

The torus $\operatorname{Fr} N$ does not intersect $\operatorname{Int} D$ so one can attach B onto N along E and obtain N^*. Then the number of components of $\operatorname{Fr} N^* \cap \operatorname{Fr} M$ is less than the number of components of $\operatorname{Fr} N \cup \operatorname{Fr} M$. Now we conclude the proof using the inductive hypothesis on solid tori M and N^* and the number $\eta'/2$ in place of η.

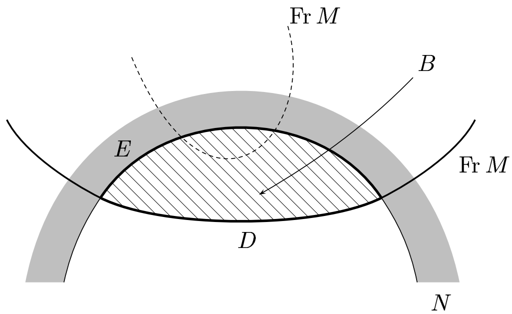

FIGURE 3. Outer disk D with respect to N.

As a result we obtain (at most) two disjoint solid tori. Finally using Lemma 7 we cut slightly, say η', enlarged disk B away from these two solid tori.

The case of solid torus. This is the remaining case when none of components of $\operatorname{Fr} M \cap \operatorname{Fr} N$ bounds a 2-disk on $\operatorname{Fr} M$ or $\operatorname{Fr} N$. There exist two components, say J_1 and J_2, which bound some annulus K on $\operatorname{Fr} N$ whose interior does not intersect $\operatorname{Int} M$. The loops J_1 and J_2 bound some annulus K' on $\operatorname{Fr} M$ and $K \cup K'$ is the boundary of some solid torus which lies entirely in M. Then we cut M along K to obtain two disjoint solid tori. One of them lies in $\operatorname{Int} N$ and it can be disposed. We denote the other one, which lies in $\mathbb{E}^3 \setminus N$, by M^*. As $\operatorname{Fr} M^* \cap \operatorname{Fr} N$ has less components than $\operatorname{Fr} M \cap \operatorname{Fr} N$, we conclude the proof by induction on M^* and N. $\square$

We were left to prove the following lemma

Lemma 7. *Let* $\mu(X) > 0$ *for a given Cantor set* $X \subset \mathbb{E}^3$. *Then for every solid torus* $T \subset \mathbb{E}^3$ *and every* 3*-ball* $B \subset \mathbb{E}^3$*, such that* $X \subset \operatorname{Int} T \setminus B$*,* $B \not\subset T$*,* $\operatorname{Fr} B \pitchfork \operatorname{Fr} T$ *and* $\operatorname{diam}(T \cup B) < \mu(X)$*, there exists a solid torus* $T' \subset T \setminus B$ *which contains* X *in its interior.*

Proof. The proof will be similar to the proof of preceding lemma. We induct on the number of components of $\operatorname{Fr} T \cap \operatorname{Fr} B$. Case $\operatorname{Fr} T \cap \operatorname{Fr} B = \varnothing$ is obvious.

If $\operatorname{Fr} T \cap \operatorname{Fr} B$ is connected, then this loop bounds two 2-disks on $\operatorname{Fr} B$ and the interior of one of them, denoted by D, lies in $\operatorname{Int} T$. As $\operatorname{diam}(T \cup B) < \mu(X)$ we may cut T along D to obtain the required torus $T' \subset T$ and some 3-ball which can be disposed.

If $\operatorname{Fr} T \cap \operatorname{Fr} B$ has at least two components we choose the innermost of them, with respect to $\operatorname{Fr} B$, and denote it by J. The loop J bounds some 2-disk $D \subset \operatorname{Fr} B$ such that $\operatorname{Int} D \cap \operatorname{Fr} T = \varnothing$. The loop J bounds some 2-disk E on $\operatorname{Fr} T$. Let B' be a 3-ball with boundary $D \cup E$. We distinguish two cases

- If $\operatorname{Int} D \subset \operatorname{Int} T$ we cut B' out of T and repeat the procedure with diminished torus T and disk B, see Figure 4.

- If $\operatorname{Int} D \subset \mathbb{E}^3 \setminus T$ then $\operatorname{Int} B' \subset \mathbb{E}^3 \setminus T$, see Figure 5. The intersection $\operatorname{Int} E \cap \operatorname{Fr} B$ may not be void so one has to push $\operatorname{Fr} B$ out of $\operatorname{Int} B'$ into that part of slightly thickened disk B' which lies in $\operatorname{Int} T$.

Finally we distinguish two subcases

If $\operatorname{Int} E \subset B$ we cut slightly enlarged disk B' out of B and repeat the procedure with torus T and diminished disk B.

If $\operatorname{Int} E \subset \mathbb{E}^3 \setminus B$ we attach B' onto B and repeat the procedure with torus T and enlarged disk B. (Note that $\operatorname{diam}(T \cup B)$ remains the same.) ◻

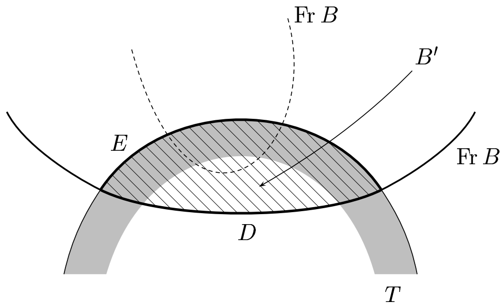

FIGURE 4. Inner disk D with respect to T.

Now we can state the main theorem for Cantor sets having local genus equal to 1.

Theorem 8. *Let $\mu(X) > 0$ for a given Cantor set $X \subset \mathbb{E}^3$. If $g_x(X) = 1$ for every point $x \in X$ then $g(X) = 1$.*

Proof. Denote $\mu(X)$ simply by μ and fix $\varepsilon > 0$. We will find a finite collection of disjoint small tori whose interiors cover X.

Using the assumption that $g_x(X) = 1$ for every point x of a compact set X there exists a finite collection $\mathcal{T} = \{T_i\}_{i=1}^m$ of tori such that

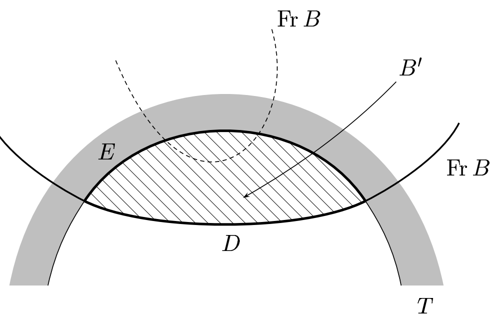

FIGURE 5. Outer disk D with respect to T.

diam $(T_i) < \min\{\varepsilon, < \mu/2\}$ and $\operatorname{Fr} T_i \cap X = \varnothing$ for every $i = 1, 2, \ldots, m$. We may also assume that boundaries of these tori intersect transversally.

We assign the number $c(\mathcal{T}) = \sum_{1 \leq i < j \leq m} c_{i,j}$ to the cover $\mathcal{T}$ where

$$c_{i,j} = \begin{cases} 0 & \text{if } \operatorname{Fr} T_i \cap \operatorname{Fr} T_j = \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

If $c_{i,j} = 0$ for every i and j the tori are disjoint and $\mathcal{T}$ is the collection we are looking for. Otherwise we define

$$\eta := \min\left\{ \frac{\varepsilon}{2(m-1)}, \min\{\operatorname{dist}(T_i, T_j);\ T_i \cap T_j = \varnothing\} \right\}$$

and pick the least pair of indexes (i, j), $i < j$, such that $c_{i,j} = 1$. Using Lemma 6 for the pair of tori $M := T_i$ and $N := T_j$ with control η we replace the tori T_i in T_j with disjoint T_i' in T_j' to obtain a new cover $\mathcal{T}'$. The number η was chosen appropriately to assure that for every $k \neq i, j$ we have: $T_i' \cap T_k = \varnothing$ if $T_i \cap T_k = \varnothing$ and $T_j' \cap T_k = \varnothing$ if $T_j \cap T_k = \varnothing$. Therefore $c(\mathcal{T}') < c(\mathcal{T})$ and we repeat the procedure with new cover $\mathcal{T}'$. The diameters of tori T_i' in T_j' have increased at most by $\varepsilon/2(m-1)$. The procedure must stop after at most $m(m-1)/2$ steps so the diameters of components increase at most to 2ε as every torus is involved in the procedure at most $m - 1$ times. □

As a trivial consequence of the preceding theorem we obtain

Corollary 9. *Let $X \subset \mathbb{E}^3$ be a nonsplittable Cantor set. If $g_x(X) = 1$ for every point $x \in X$ then $g(X) = 1$.*

We say that the Cantor set X is *locally nonsplittable* if, for every point $x \in X$, there exists a neighborhood $U \subset \mathbb{E}^3$ of x such that $X \cap U$ is a nonsplittable Cantor set. Therefore

Corollary 9. *Every locally nonsplittable and locally toroidal Cantor set is toroidal.*

6. Genus of the union of Cantor sets. If the Cantor sets X and Y are disjoint we have $g(X \cup Y) = \max\{g(X), g(Y)\}$. A tame Cantor set behaves nicely with respect to the genus as we have

Theorem 11. *Let $X \subset \mathbb{E}^3$ be a tame Cantor set. Then $g(X \cup Y) = g(Y)$ for every Cantor set $Y \subset \mathbb{E}^3$.*

Proof. The estimation $g(Y) \leq g(X \cup Y)$ is obvious. Now pick an arbitrarily defining sequence (M_i) for Y. We will prove that for every index i there exists a manifold N_i which contains $X \cup Y$ in its interior such that $\operatorname{diam} N_i \leq 2 \operatorname{diam} M_i$ and $g(N_i) = g(M_i)$.

Let $\varepsilon = \operatorname{dist}(Y, \operatorname{Fr} M_i)/2$. As $X \subset \mathbb{E}^3$ is a tame Cantor set it can be pushed off the 2-manifold $\operatorname{Fr} M_i$ by some ε-move h. Hence $h^{-1}(M_i)$ is a cube with handles which contains Y in its interior and $\operatorname{Fr}(h^{-1}(M_i)) \cap X = \varnothing$. The manifold N_i is therefore $h^{-1}(M_i)$ union some disjoint small 3-balls which cover a tame Cantor set $X \setminus h^{-1}(M_i)$. □

As in [**5**] we denote by $\mathrm{T}(X)$ the set of all such points x of the Cantor set X, where X is locally tame at x.

Theorem 12. *Let $X, Y \subset \mathbb{E}^3$ be Cantor sets. If $X \cap Y \subset \mathrm{T}(X) \cap \mathrm{T}(Y)$, then $g(X \cup Y) = \max\{g(X), g(Y)\}$.*

Proof. By [**5**] the set $\mathrm{T}(X)$ is open in X and $\mathrm{T}(Y)$ is open in Y. By assumption of the theorem we have

$$X \cap Y \subset \mathrm{T}(X) \cap \mathrm{T}(Y) \subset X \cap Y$$

and hence $X \cap Y = \mathrm{T}(X) \cap \mathrm{T}(Y)$. Then the Cantor sets $X' = X \setminus (\mathrm{T}(X) \cap \mathrm{T}(Y))$, $Y' = Y \setminus (\mathrm{T}(X) \cap \mathrm{T}(Y))$ and $X \cap Y$ are pairwise disjoint. Because of $X \cap Y = \mathrm{T}(X) \cap \mathrm{T}(Y)$ this set is tame and hence

$$g(X \cup Y) = g(X' \cup Y') = \max\{g(X'), g(Y')\} = \max\{g(X), g(Y)\},$$

using $g(X) = g(X')$ and $g(Y) = g(Y')$. □

Theorem 13. *Let $X, Y \subset \mathbb{E}^3$ be nondisjoint Cantor sets and $a \in X \cap Y$ a point that there exists a 3-ball B and a 2-disk $D \subset B$ such that*

1. *$a \in \operatorname{Int} B$, $\operatorname{Fr} D = D \cap \operatorname{Fr} B$, $D \cap (X \cup Y) = \{a\}$ and*

2. *We have* $X \cap B \subset B_X \cup \{a\}$ *and* $Y \cap B \subset B_Y \cup \{a\}$ *where* B_X *and* B_Y *are the components of* $B \setminus D$.
Then $g_a(X \cup Y) = g_a(X) + g_a(Y)$.

Proof. Let us prove that $g_a(X \cup Y) \leq g_a(X) + g_a(Y)$. There exists such defining sequences (M_i) for X and (N_i) for Y that $g_a(X; (M_i)) = g_a(X)$ in $g_a(Y; (N_i)) = g_a(Y)$. Let i be so large that the component M of M_i and the component N of N_i which contains a both lie $\operatorname{Int} B$. We may assume that $\operatorname{Fr} M$ and $\operatorname{Fr} N$ intersect D transversally. Then $\operatorname{Fr} M \cap D$ consists of finitely many pairwise disjoint circles and by cut and paste techniques as in the proof of Theorem 4 one can assume that $M \cap D$ is a 2-disk containing a in its interior. As $X \cap B_Y = \varnothing$, the manifold $M \cap \overline{B_X}$ is a cube with at most $g(\operatorname{Fr} M)$ handles and its boundary intersects $X \cup Y$ only in point a. Similarly one can modify N so that $N \cap \overline{B_Y}$ is a cube with at most $g(\operatorname{Fr} N)$ handles and its boundary intersects $X \cup Y$ only in point a. If we modify M and N carefully we also obtain $M \cap D = N \cap D$. Then $Q = (M \cap \overline{B_X}) \cup (N \cap \overline{B_Y})$ is a cube with at most $g(\operatorname{Fr} M) + g(\operatorname{Fr} N)$ handles, $a \in \operatorname{Int} Q$ and $\operatorname{Fr} Q \cap (X \cup Y) = \varnothing$. Hence $g_a(X \cup Y) \leq g_a(X) + g_a(Y)$.

For the proof of $g_a(X \cup Y) \geq g_a(X) + g_a(Y)$ we take such defining sequence (Q_i) $X \cup)$ that $g_a(X \cup Y; (Q_i)_i) = g_a(X \cup Y)$. As in the first part of the proof we modify Q_i so that $D \cap Q_i$ is connected. Now we cut Q_i along D and thicken the components. We get the manifolds Q_i^X in Q_i^Y for which $\operatorname{Fr}(Q_i^X) \cap X = \operatorname{Fr}(Q_i^Y) \cap Y = \varnothing$ and $x \in Q_i^X \cap Q_i^Y$. We may assume that 2-disk B is so small that $g(M) \geq g_a(X)$ for every cube with handles $M \subset \operatorname{Int} B$ which contains a and $g(N) \geq g_a(Y)$ every cube with handles $N \subset \operatorname{Int} B$ which contains a. Hence $g(Q_i) = g(Q_i^X) + g(Q_i^Y) \geq g_a(X) + g_a(Y)$ and therefore $g_a(X \cup Y) \geq g_a(X) + g_a(Y)$. □

Remark. Using the preceding theorem one can alternatively prove the existence of the Cantor set of given genus.

Summarizing the above theorems one may conjecture:

Conjecture 3. *For arbitrary Cantor sets $X, Y \subset \mathbb{E}^3$ we have*

$$\max\{g(X), g(Y)\} \leq g(X \cup Y) \leq g(X) + g(Y). \tag{3}$$

Using (1) we easily prove the left inequality above. But the right inequality above is not true in general. We will briefly explain the defining sequences for such Cantor sets.

Let X and Y be self-similar Cantor sets given by defining sequences (M_i) in (N_i) which are symmetric with respect to $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$, see Figure 6. The plane $\mathbb{E}^2 \times \{0\} \subset \mathbb{E}^3$ contains equators of all 3-balls.

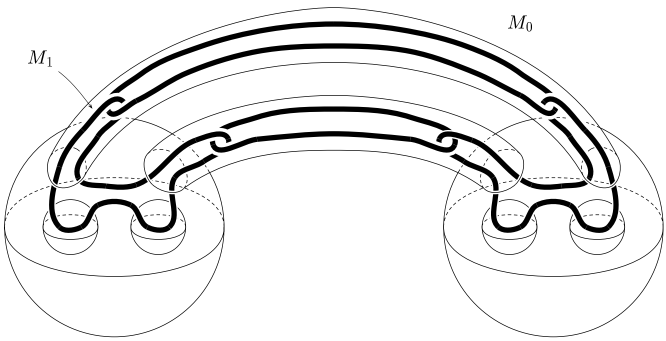

FIGURE 6. Example of $g(X \cup Y) = g(X) + g(Y) + 1$.

We have $X \cap Y \subset \mathbb{E}^2 \times \{0\}$ hence the (Cantor) set $X \cap Y$ is tame. Obviously $g(X) = g(Y) = 1$ and one can prove that $g_a(X \cup Y) = 3$ for every $a \in X \cap Y$.

Hence the new conjecture is

Conjecture 4. *If the intersection of Cantor sets $X \subset \mathbb{E}^3$ and $Y \subset \mathbb{E}^3$ is a tame (Cantor) set, we have*

$$g(X \cup Y) \leq g(X) + g(Y) + 1.$$

The author believes that in general genus of the union of Cantor is not related to $g(X) + g(Y)$, more precisely

Conjecture 5. *For every $r \in \mathbb{N}$ there exist Cantor sets X and Y, such that*

$$g(X \cup Y) \geq g(X) + g(Y) + r.$$

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