MINIMAL NUMBER OF TORI IN GEOMETRIC SELF-SIMILAR ANTOINE CANTOR SETS

MATJAŽ ŽELJKO

Abstract. In a defining sequence of a standard Antoine Cantor every topological tori has an embedded chain of 4 smaller topological tori. If we require the tori to be geometric tori symetrically embedded in the torus of previous stage, the embedded chain must have at least 20 terms.

1. Introduction

A Cantor set X is *self-similar* if there exists a defining sequence (M_i) such that $X = \bigcup_i M_i$ and for every i and every component M of M_i the pair $(M, M \cap M_{i+1})$ is homeomorphic to the pair (M_1, M_2). Self-similar Cantor sets are easy to describe and lots of the known examples of Cantor sets are self-similar.

The first known example of a wild cantor set, i.e. Antoine necklace (or Antoine Cantor set) is defined as a self-simlar Cantor set with M_1 as torus and M_2 as a chain of four simply linked tori inside M_1. Bing [1] has proven that Antoine construction can be done even with only two tori in M_2. (Note that the "obvious" construction with two tori does not yield to a Cantor set since diameters of components do not tend to 0.)

A *simple Antoine Cantor set* X in $\mathbb{R}^3$ is a Cantor set satisfying the following conditions.

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(1) X has a defining sequence (M_i) where each term consists of the union of a finite number of pairwise disjoint standard unknotted solid tori in $\mathbb{R}^3$ and M_1 is a single solid torus.
(2) The components of M_i, $i \geq 2$, can be ordered in a sequence $M_{i,1}$, $M_{i,2}$, $\ldots$, M_{i,n_i} where $M_{i,j}$ and $M_{i,k}$ are linked if and only if $j - k \equiv \pm 1 \pmod{n_i}$.
(3) The linked chain of tori $M_{i,1}$, $M_{i,2}$, $\ldots$, M_{i,n_i} has a winding number equal 1 in the torus at the previous stage that contains it.

Let $n > 2$ be a positive integer. IF for every component M of M_i the set $M \cap M_{i+1}$ has exactly n componets, we say that Antoine Cantor set X is of order n. If there exists geometric similarities $g_j \colon \mathbb{R}^3 \to \mathbb{R}^3$, $j = 1, 2, \ldots, n$, such that for every component M of M_i the components of $M \cap M_{i+1}$ are $g_j(M_i)$, $j = 1, 2, \ldots, n$, then the set X is called *geometric self-similar*.

Geometric self-similar Antoine Cantor set is *a regular* if for every component M of M_i the tori in $M \cap M_{i+1}$ form a regular chain of tori; i.e. core circles for every two consecutive tori lie in perpendicular planes and centres of tori form a regular n-gon.

Theorem 1. *If X is a regular geometric self-similar Antoine Cantor set then the order of X is at least* 20.

In [2] Malešič and Repovš construct a regular geometric self-similar Antoine Cantor set in $\mathbb{R}^3$ of order 60 that is Lipschitz homogeneously embedded. Using the theorem above one can replace the Cantor set in the above paper by the another regular geometric self-similar Antoine Cantor set of order 20.

2. The Proof

Let A', B and C' be the centers of three consecutive tori. Then A', B and C' are consectutive vertices of a regular n-gon, where central angle φ equals to $\frac{2\pi}{n}$.

The outer torus is given by two radii: R and r. Let $k > 0$ be a coefficient of similitude. Then the smalest torus has diameter $2k(R+r)$ in order to fit in the larger one, the condition

$$2k(R+r) < 2r \tag{2.1}$$

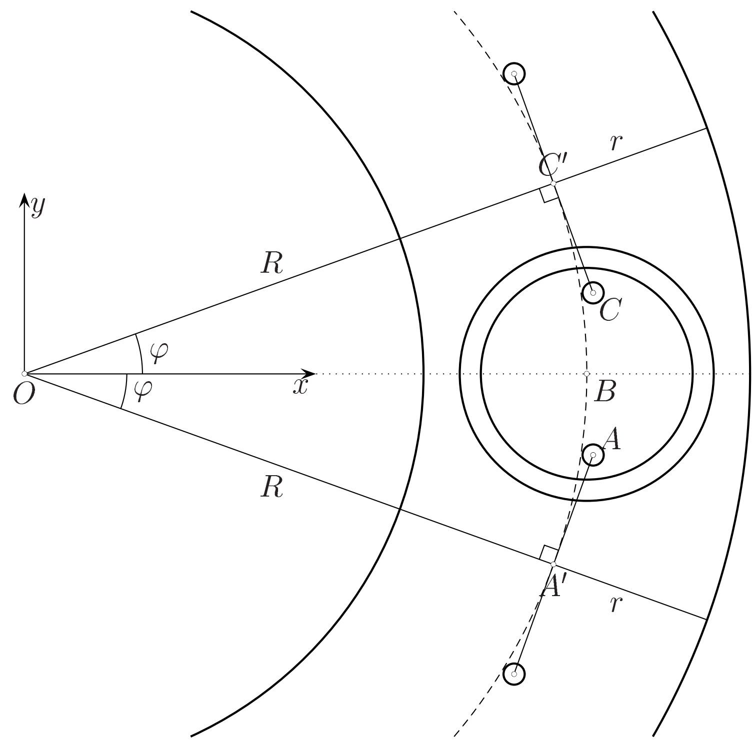
has to be satisfied.

Let us calculate some coordinates. By definition we have $O(0,0)$ and $B(R,0)$. By simlitude we have $|AA'| = |BB'| = kR$ and therefore

$$A(R\cos\varphi + kR\sin\varphi, -R\sin\varphi + kR\cos\varphi)$$

and

$$C(R\cos\varphi + kR\sin\varphi, R\sin\varphi - kR\cos\varphi).$$



The circles centered at A and C have radii kr and must not intersect; hence

$$R\sin\varphi - kR\cos\varphi > kr. \tag{2.2}$$

In order to get proper linking one we require

$$|BC| + rk < k(R - r). \tag{2.3}$$

Let us intruduce $\tau = \frac{r}{R}$ (obviously $0 < \tau < 1$) and simplify the conditions. The inequality (2.1) becomes

$$k < \frac{\tau}{1 + \tau} \tag{2.4}$$

The inequality (2.2) becomes

$$\sin\varphi - k\cos\varphi > k\tau \tag{2.5}$$

As

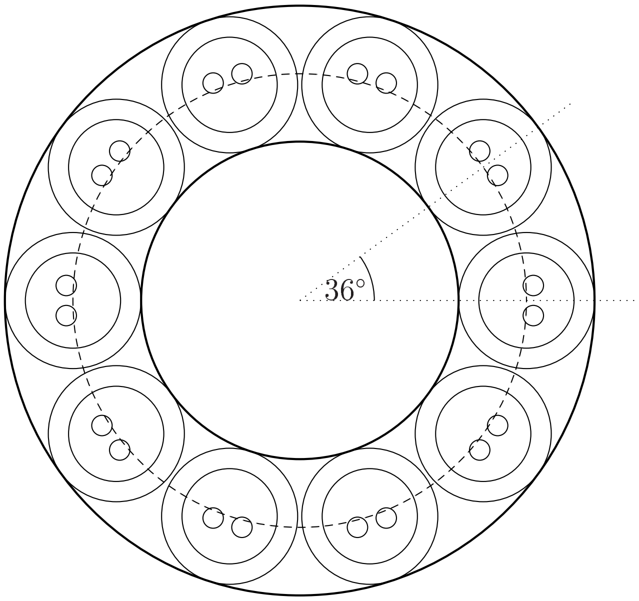
$$|BC|^2 = (R\cos\varphi + kR\sin\varphi - R)^2 + (R\sin\varphi - kR\cos\varphi)^2$$

the inequality (2.3) becomes

$$\sqrt{(\cos\varphi + k\sin\varphi - 1)^2 + (\sin\varphi - k\cos\varphi)^2} < k(1 - 2\tau). \tag{2.6}$$

The number of links has to be an even integer, so φ is of form $\varphi = \frac{2\pi}{n}$, where $n \geq 4$ is an even interger. Using numerical methods one can easily find that the **largest** φ satisfying equations (2.4), (2.5) and (2.6) equals to $\frac{\pi}{10}$ (where $\tau = \frac{3}{10}$ and $k = \frac{3}{13}$). Hence the number of links in this chain is 20. (Although number n has to be even, no odd n smaller than 20 satisfies the equations above.)

The cross-section of the chain with $n = 20$ is drawn on the picture below.



It is important to notice that the centres of small tori lie on the core circle of the outer torus thus allowing to slightly rotate small tori and produce a twisted chain.

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References

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Institute of Mathematics, Physics and Mechanics,, University of Ljubljana, Jadranska 19, P. O. Box 2964, Ljubljana, Slovenia
E-mail address: matjaz.zeljko@fmf.uni-lj.si