Monatsh. Math. 145, 239–245 (2005)
DOI 10.1007/s00605-004-0294-8

Monatshefte für
Mathematik
Printed in Austria

On Unions and Intersections of Simply Connected Planar Sets

By

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Received October 14, 2003; accepted in revised form November 3, 2004
Published online June 23, 2005

Abstract. We construct several simple examples of planar compacta which show that without additional conditions, a theorem of Breen and a direct generalization of the Seifert-van Kampen theorem fail. We give answers to two conjectures of Bogatyi and a partial solution to his third conjecture. We give a counterexample to a statement in the classical survey paper by Danzer–Grünbaum–Klee, related to Molnár's result on intersections of simply connected planar sets.

2000 Mathematics Subject Classification: 54F15, 55N10, 54D05
Key words: Simply connected planar sets, planar continua, Helly-type theorem, Molnár's theorem, Seifert-van Kampen theorem

1. Introduction

A space X is called *simply connected* if it is path-connected and its fundamental group is trivial. The fundamental group of a planar space X is trivial if and only if for every Jordan curve $\mathscr{J} \subset X$ and every point $q \in X \backslash \mathscr{J}$ in the bounded region determined by $\mathscr{J}$, q belongs to X (see [15, p. 107, Proposition 2.51] and [10, Chapter 10, §61, II, Theorem 5]) or equivalently, no Jordan curve in X is a retract of X.

There exist two simply connected planar sets with simply connected intersection but non-simply connected union [9, p. 284, Fig. 1]:

Clearly, the spaces X_0, X_1 and $X_0 \cap X_1$ are simply connected but the union $X_0 \cup X_1$ contains a topological circle which is its retract and therefore the union is not simply connected. We generalize this theorem to an arbitrary finite number of simply connected sets as follows:

Theorem 1.1. *For an arbitrary natural number n there exists a family $\mathscr{F} = \{X_i\}_{i=0}^{n}$ of simply connected compact subsets of $\mathbb{R}^2$ such that*:

(1) *The unions $\bigcup_{k=0}^{l} X_{i_k}$ for all $l < n$ and the intersections $\bigcap_{k=0}^{l} X_{i_k}$ for all $l \leqslant n$ are simply connected.*

(2) *The intersection $\bigcap_{i=0}^{n} X_i$ is nonempty.*

(3) *The union $\bigcup_{i=0}^{n} X_i$ is not simply connected.*

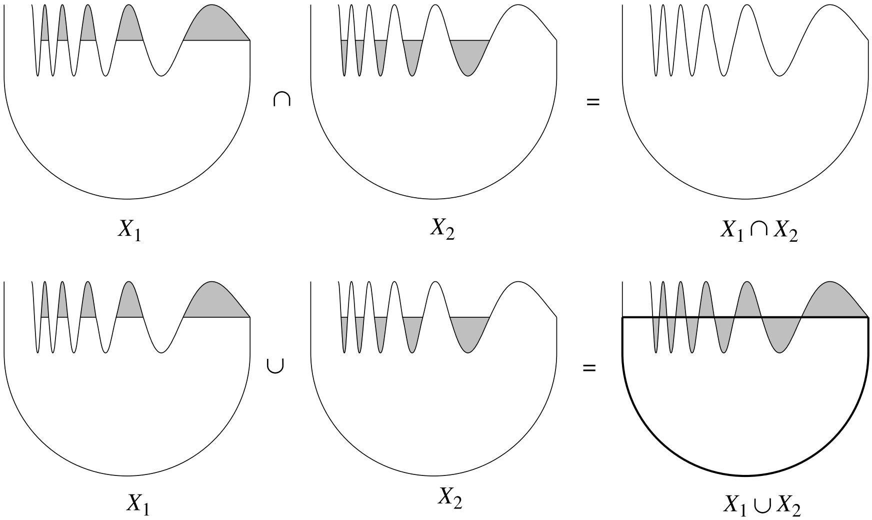

Figure 1

For $n = 2$, Theorem 1.1 gives a counterexample to Theorem 1 from [3; p. 112] (even though our definition of "simply connected" is somewhat different from the definition in Breen's paper) and to Conjecture 1 of [2; p. 398] (see also Section 4, Conjecture B(1) below).

The answer to Conjecture 4 of [2, p. 400] (see Section 4, Conjecture C below) follows from our Theorem 1.1 for $n \geqslant 3$. These planar compacta are simple examples which show that without some additional conditions the direct generalization of Seifert-van Kampen theorem fails (the standard proofs of this fact are difficult – see e.g. [4; pp. 314–315] and the references there).

The affirmative answer to Conjecture 3 of [2, p. 400]) (see Section 4, Conjecture B(2) below) follows from our second main result:

Theorem 1.2. *If in a family of planar simply connected compact or open subspaces* $\mathscr{F} = \{X_i\}_{i=0}^{n} \subset \mathbb{R}^2$ *the intersection of every two members is path-connected and the intersection of every three members is nonempty, then the intersection of all members* $\bigcap_{i=0}^{n} X_i$ *is nonempty.*

2. Proof of Theorem 1.1

We will show how the corresponding example can be constructed for every $n \in \mathbb{N}$. Fix the number n. Consider the closed topologist's sine curve in the plane $\mathbb{R}^2$:

$$T = \left\{ (x, y) \in \mathbb{R}^2 | 0 < x \leqslant 1, \quad y = \sin\left(\frac{3\pi}{2x}\right) \right\} \cup (\{0\} \times [-1, 1]).$$

Define for every $k \in \mathbb{N}$ the set A_k:

$$A_k = \left\{ (x, y) \in \mathbb{R}^2 | \frac{3}{4k+3} \leqslant x \leqslant \frac{3}{4k-1} \quad \text{and} \quad -1 \leqslant y \leqslant \sin\left(\frac{3\pi}{2x}\right) \right\}.$$

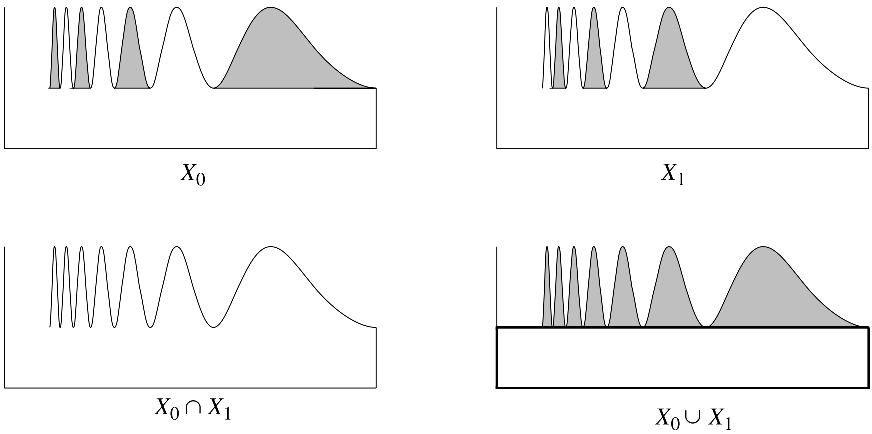

Figure 2

Let L be any arc connecting the points $(0,-1)$ and $(1,-1)$, which does not intersect $T \cup \bigcup_{k=1}^{\infty} A_k$ in other points.

Let

$$X_i = T \cup L \cup \bigcup_{k=1}^{\infty} A_{(k-1)(n+1)+i+1},$$

for $i \in \{0, 1, 2, \ldots, n\}$.

Figure 2 depicts the family $\{X_i\}_{i=0}^{n}$ for $n = 1$.

All proper unions $\bigcup_{k=0}^{l} X_{i_k}$ and the intersections $\bigcap_{k=0}^{l} X_{i_k}$ for $l \leqslant n$ are simply connected nonempty sets. The union $\bigcup_{i=0}^{n} X_i$ is not simply connected since it contains the topological circle $L \cup \alpha$ which is its retract (α is the arc $[0,1] \times \{-1\}$). □

3. Proof of Theorem 1.2

We shall need the following lemma which is a direct consequence of Sperner's lemma (see e.g. [1, p. 161]) and a theorem of Lassonde (see [11, Théorème 2 on p. 574]):

Lemma 3.1. *Let Δ^n be an n-simplex with vertices $e_0, e_1, \ldots, e_n$. Let $\{A_i\}_{i=0}^{n}$ be a closed or an open covering of Δ^n, satisfying the condition that every face $[e_{i_0}, e_{i_1}, \ldots, e_{i_m}]$ of Δ^n is contained in $\cup_{j=0}^{m} A_{i_j}$. Then the intersection of all sets A_i is nonempty.*

Proof of Theorem 1.2. Choose a point x_i in every X_i, a point x_{ij} in every $X_i \cap X_j$ and a point x_{ijk} in every $X_i \cap X_j \cap X_k$. Consider an n-dimensional simplex Δ_n with the vertices $e_0, e_1, \ldots, e_n$. Denote the m-dimensional skeleton of Δ_n by Δ_n^m. Construct the mappings $f_m : \Delta_n^m \to \bigcup_{i=0}^{n} X_i$ inductively:

(1) Begin by setting $f_0(e_i) = x_i$.

(2) Since the spaces X_i are path-connected and the intersections $X_i \cap X_j$ contain the point x_{ij}, there exists a path connecting x_i and x_{ij} in X_i. Therefore there

exists a mapping $f_1 : \Delta_n^1 \to \bigcup_{i=0}^n X_i$ which is the extension of f_0, which maps the barycenter e_{ij} of the $[e_i, e_j]$ to x_{ij} and which satisfies the condition that $f_1([e_i, e_{ij}]) \subset X_i$.

(3) Construct the extension $f_2 : \Delta_n^2 \to \bigcup_{i=0}^n X_i$ of f_1 as follows: Consider a 2-dimensional face $[e_i, e_j, e_k]$ of the simplex Δ_n. Let e_{ijk} be its barycenter and let $f_2(e_{ijk}) = x_{ijk}$. Since every member and the intersection of every two members of $\{X_i\}_{i=0}^n$ are path-connected, there exists an extension of f_1 onto the 1-dimensional skeleton of the barycentric subdivision of the polyhedron Δ_n^2 which maps $[e_{ijk}, e_i]$ to X_i and the edges $[e_{ijk}, e_{ij}]$ to $X_i \cap X_j$. Since X_i is simply connected, there exists an extension $f_2 : \Delta_n^2 \to \bigcup_{i=0}^n X_i$ such that $f_2([e_i, e_j, e_k]) \subset X_i \cup X_j \cup X_k$.

(4) Suppose now that for some $m < n$ the mapping f_m has already been constructed. Consider any $(m+1)$-dimensional simplex $[e_{i_0}, e_{i_1}, \ldots, e_{i_{m+1}}]$ of Δ_n^{m+1}. We already have a mapping $f_m : \partial[e_{i_0}, e_{i_1}, \ldots, e_{i_{m+1}}] \to \bigcup_{j=0}^{m+1} X_{i_j}$ of the boundary $\partial[e_{i_0}, e_{i_1}, \ldots, e_{i_{m+1}}]$. By Zastrow's theorem on asphericity of planar subsets [5, 16], this mapping can be extended to the $(m+1)$-dimensional simplex $[e_{i_0}, e_{i_1}, \ldots, e_{i_{m+1}}]$. In this manner we obtain the mapping $f_{m+1} : \Delta_n^{m+1} \to \bigcup_{i=0}^n X_i$.

We can conclude by induction that there exists a mapping $f : \Delta_n \to \bigcup_{i=0}^n X_i$ such that the image of every face $[e_{i_0}, e_{i_1}, \ldots, e_{i_m}]$ is contained in $\bigcup_{j=0}^m X_{i_j}$. It follows that $[e_{i_0}, e_{i_1}, \ldots, e_{i_m}] \subset \bigcup_{j=0}^m f^{-1}(X_{i_j})$. Since the family $\mathscr{F}$ consists of closed or open sets it follows by Lemma 3.1 that $\bigcap_{i=0}^n f^{-1}(X_i) \neq \emptyset$ and therefore $\bigcap_{i=0}^n X_i \neq \emptyset$ $\square$.

Remark. The condition of path-connectedness in Theorem 1.2 cannot be replaced by the condition of connectedness (see Section 5).

4. On Conjectures of Bogatyi

Bogatyi [2] stated (among others) the following three conjectures (recall the characterization of simple connectivity in the plane, given in our introduction):

Conjecture A ([2, p. 398, Conjecture 1]). *If the intersection $A_0 \cap A_1$ of two planar simply connected continua A_0 and A_1 is path-connected then the union $A_0 \cup A_1$ is simply connected.*

Conjecture B ([2, p. 400, Conjecture 3]). *If in a finite family of planar simply connected continua the intersection of every two continua is path-connected and every three continua have a common point then the following hold*:

(1) *The union of all continua is simply connected.*
(2) *The intersection of all continua is nonempty.*
(3) *The intersection of all continua is path-connected.*

Conjecture C ([2, p. 400, Conjecture 4]). *If in a finite family of planar simply connected continua the union of every two and every three continua is simply connected then the union of all continua of the family is simply connected.*

A counterexample to Conjectures A and B(1) for $n = 1$ and for $n = 2$ respectively, follows from our Theorem 1.1. Conjecture B(2) follows directly from our Theorem 1.2. Conjecture B(3) is equivalent to a statement of Eckhoff (see next

Section 5). A counterexample to Conjecture C follows by our Theorem 1.1, applied for $n \geqslant 3$.

5. On Statements by Danzer–Grünbaum–Klee and Eckhoff

The space is called *acyclic* if all of its reduced singular homology groups are trivial. It was asserted by Danzer–Grünbaum–Klee in [6; p. 125] that for the plane $\mathbb{R}^2$ Molnár [12, 13] established the following improvement of Helly's Topological Theorem:

Statement 5.1. *A family of at least three simply connected compact sets in* $\mathbb{R}^2$ *has a nonempty simply connected intersection, provided that each two of its members have connected intersection and each three members have a nonempty intersection.*

We now give an example which shows that this assertion is *incorrect*: Let L_1 and L_2 be any arcs in the plane connecting the points $(0, -1)$ and $(1, -1)$ such that $L_1 \cap L_2 = (L_1 \cup L_2) \cap T = \{(0, -1),\ (1, -1)\}$, see Section 1 for the definition of the set T. Let $X_i = L_i \cup T$ for $i = 1, 2$, and X_3 be a closed topological disk in the plane generated by $L_1 \cup L_2$. Figure 3 below depicts these spaces.

Clearly all sets $X_i \cap X_j$ are connected. Nevertheless, $X_1 \cap X_2 \cap X_3$ is *not connected*. If we subdivide the set X_3 into two topological disks X_3' and X_4', by a "vertical" line in the middle, we get the family $\{X_1, X_2, X_3', X_4'\}$ which satisfies the hypotheses of Statement 5.1 above. However, $X_1 \cap X_2 \cap X_3' \cap X_4'$ is the *empty set*. □

In [7; p. 402] it was stated by Eckhoff that Helly [8] established the following result:

Statement 5.2. *Let* $\mathscr{K}$ *be a finite family of closed sets in* $\mathbb{R}^d$ *such that the intersection of every* k *members of* $\mathscr{K}$ *is acyclic for* $k \leqslant d$ *and is nonempty for* $k = d + 1$. *Then* $\cap\mathscr{K}$ *is acyclic.*

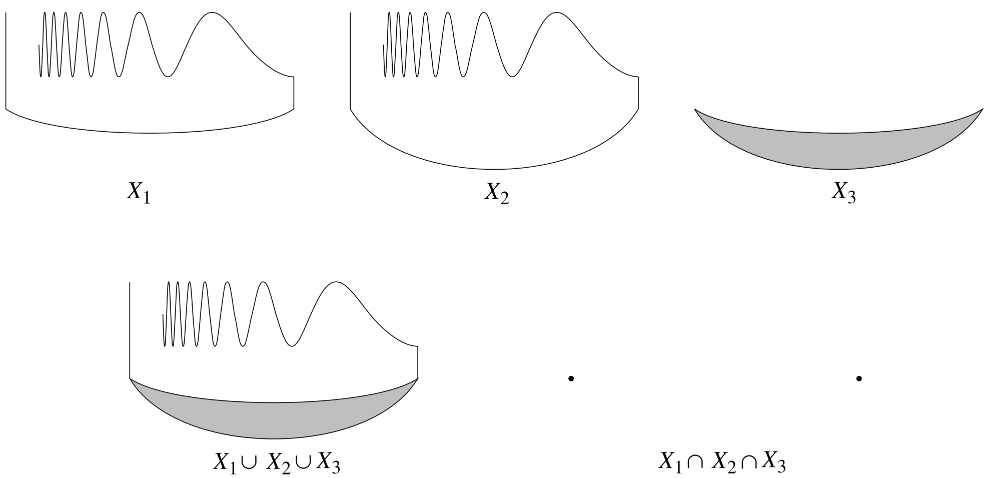

Figure 3

We were unable to find any proof of Statement 5.2 in the literature (note that Helly [8] used Vietoris homology). However, we will now prove that Conjecture B(3) is equivalent to the Statement 5.2 for $d = 2$.

Suppose that Statement 5.2 is valid and let $\mathscr{K}$ be a family of spaces satisfying the hypotheses of Conjecture B(3). Then the intersection of every two continua of the family is path-connected. Since the fundamental group of the intersection of any two simply connected planar set is trivial (consider any Jordan curve $\mathscr{J}$ in the intersection – the bounded region determined by $\mathscr{J}$ belongs to both subsets), it follows that the intersection of every two elements of the family is simply connected. By Zastrow's theorem on asphericity of planar spaces ([5, 16]) and by the Hurewicz theorem (see e.g. [14, p. 397]), every simply connected planar space is acyclic. By Statement 5.2 it now follows that the intersection of all members of the family is acyclic and therefore it is a path-connected set. Therefore Statement 5.2 implies Conjecture B(3).

Suppose now that the Conjecture B(3) is valid and assume that the hypotheses of Statement 5.2 are satisfied for some family $\mathscr{K}$. Every acyclic planar subspace has trivial fundamental group. (Otherwise as was mentioned in Section 1 there would be a Jordan curve $\mathscr{J}$ which would be its retract and therefore the 1-dimensional singular homology group would be nontrivial). It now follows that every acyclic planar set is simply connected and the family $\mathscr{K}$ satisfies the conditions of Conjecture B(3). Therefore $\bigcap \mathscr{K}$ is a path-connected set with the trivial fundamental group. Therefore $\bigcap \mathscr{K}$ is an acyclic planar set and Conjecture B(3) implies Statement 5.1 for $d = 2$. □

Acknowledgements. The second and the third author were supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia research program No. P1-0292. We thank the referee for a very thorough reading of the manuscript and several useful comments and suggestions.

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